On the Last Zero of a Spectrally Negative Lévy Process

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Research Showcase LSE

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[1] joint work with José Pedraza

Insurance

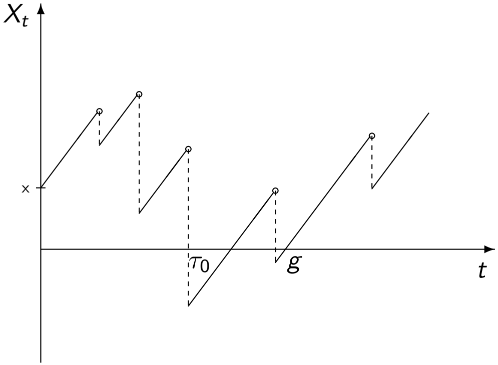
Cramér–Lundberg Process

$$X_t = x + ct - \sum_{j=1}^{N_t} Y_j,$$

where $x, c > 0$, N_t is a Poisson process with intensity $\lambda > 0$ and $\{Y_j\}_{j \geq 1}$ is a sequence of positive i.i.d random variables independent of N_t.

Two quantities of interest are the moment of ruin and the last zero of the process

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}$$
$$g = \sup\{t \geq 0 : X_t \leq 0\}$$


X_t
x
τ_0
g
t

Degradation models

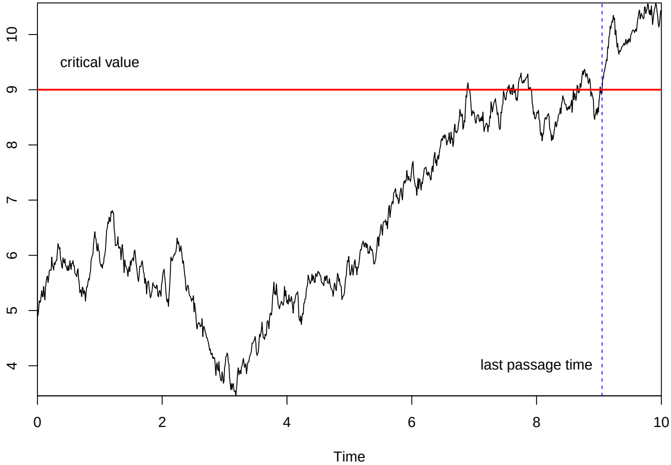
We can model the ageing of a device with $D = (D_t, t \geq 0)$ where

$$D_t = G_t + \sigma B_t$$

where $\sigma \geq 0$, $(G_t, t \geq 0)$ is a subordinator and $(B_t, t \geq 0)$ is an standard Brownian motion. Then, D is an spectrally positive Lévy process.
The failure time of the device can be defined as

$$g^* = \sup\{t > 0 : X_t \geq f_*\}$$

where f_* is a critical value.


critical value
last passage time
Time
0
2
4
6
8
10
4
5
6
7
8
9
10

Last passage times

Last passage times are random times which are not stopping times. For example, if

$$g = \sup\{t > 0 : X_t \leq 0\}$$

then we have that

$$\{g < t\} = \{X_s > 0 \text{ for all } s > t\} \in \mathcal{F}.$$

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Brownian motion: Azéma martingale

In Brownian case θ_t last hitting time of zero before time t many results are known, many linked to Azéma's martingale

$$\operatorname{sgn}(B_t)\frac{\pi}{2}\sqrt{t-\theta_t}.$$

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Cetin, U. 2012 Filtered Azéma martingales.
Dassios, A., Lim J. 2018 A variation of the Azéma martingale and drawdown options

Lévy processes

A process $X = (X_t, t \geq 0)$ is said to be a Lévy process if

- The paths of X are $\mathbb{P}$-a.s. càdlàg
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Basically: Brownian motion with jumps.

Examples

- Brownian motion
- Compound Poisson process
- Gamma process

The law of a Lévy process is characterised by the characteristic exponent,

$$\Psi(\theta) = -\log\left(\mathbb{E}(e^{i\theta X_1})\right).$$

Lévy–Khintchine Formula for Lévy processes

Exist $\sigma \geq 0$, $\mu \in \mathbb{R}$ and measure Π (Lévy measure) concentrated on $\mathbb{R} \setminus \{0\}$, with $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$, such that

$$\Psi(\theta) = i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{I}_{\{|x|<1\}})\Pi(dx)$$

for all $\theta \in \mathbb{R}$.

Lévy–Itô decomposition

$$X_t = \sigma B_t - \mu t + \int_0^t \int_{\{|x|\geq 1\}} xN(ds, dx) + \int_0^t \int_{\{|x|<1\}} x(N(ds, dx) - ds\Pi(dx))$$

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Default time can be generalised to (V, X) as its performance measure, where X can be taken to be an exponential Lévy process.

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A spectrally negative Lévy process is a Lévy processes with only negative jumps $\Pi(0,\infty) = 0$ and not monotone paths.

In this case the Laplace exponent defined as

$$\psi(\lambda) = \log\left(\mathbb{E}(e^{\lambda X_1})\right)$$

always exists and we have that $\psi'(0+) = \mathbb{E}(X_1)$. We also define the right-inverse of ψ by

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$$

Scale functions

For $q \geq 0$, $W^{(q)}$ is a continuous and strictly increasing function in $(0, \infty)$ such that $W^{(q)}(x) = 0$ for $x < 0$ and its Laplace transform is given

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q} \qquad \beta > \Phi(q),$$

where $\Phi(q)$ is the right inverse function of ψ. We also define the the function $Z^{(q)}$ as

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy.$$

Last zero and exercursions

Let X be a spectrally negative Lévy process drifting to infinity. Let $t \geq 0$ and $x \in \mathbb{R}$, we define as $g_t^{(x)}$ as the last time that the process is below x before time t, i.e.,

$$g_t^{(x)} = \sup\{0 \leq s \leq t : X_s \leq x\},$$

with the convention $\sup \emptyset = 0$. We simply denote $g_t := g_t^{(0)}$ for all $t \geq 0$. We define

$$U_t := t - g_t$$

the time of the current excursion before time t above zero.

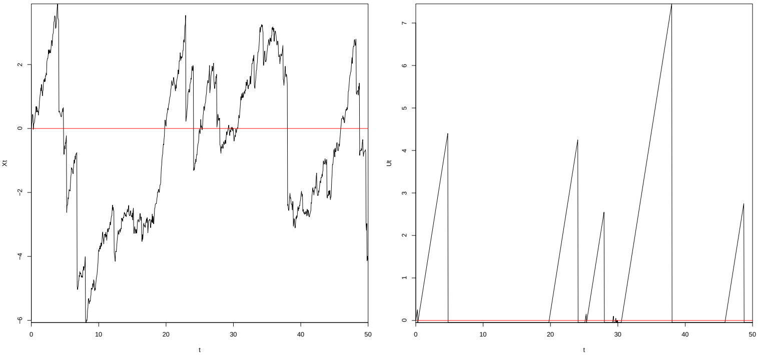

Figure: Sample path of X on the left hand side. Sample path of U_t on the right hand side.

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- For nice functions h and stopping time τ conditional expectation

$$\mathbb{E}(h(g_{\tau+s}, \tau + s, X_{\tau+s}) | \mathcal{F}_\tau) = f_s(g_\tau, \tau, X_\tau),$$

where for any $(\gamma, t, x) \in E_g$,

$$f_s(\gamma, t, x) = \mathbb{E}_x(h(\gamma, t + s, X_s)\mathbb{I}_{\{\sigma_0^- > s\}})$$

$+\mathrm{E}_x(h(g_s + t, t + s, X_s)\mathbb{I}_{\{\sigma_0^- \leq s\}})$. where
$\sigma_0^- = \inf\{t > 0 : X_t \leq 0\}$.

Itô formula

Theorem

For nice enough F we have the Itô formula for (g, t, X)

$$
\begin{aligned}
F(g_t, t, X_t) &= F(g_0, 0, X_0) + \int_0^t \frac{\partial F_g}{\partial t}(s, X_{s-})\mathbb{I}_{\{g_{s-}=s\}}\mathrm{d}s + \int_0^t \frac{\partial F}{\partial t}(g_{s-}, s, X_{s-})\mathbb{I}_{\{g_{s-}<s\}}\mathrm{d}s \\
&+ \int_0^t \frac{\partial F}{\partial x}(g_{s-}, s, X_{s-})\mathrm{d}X_s + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2 F}{\partial x^2}(g_s, s, X_s)\mathrm{d}s \\
&+ \int_{[0,t]} \int_{(-\infty,0)} \left[F(g_s, s, X_{s-} + y) - F(g_{s-}, s, X_{s-}) - y\frac{\partial F}{\partial x}(g_{s-}, s, X_{s-})\right] N(\mathrm{d}s \times \mathrm{d}y)
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= {} & F(g_0,0,X_0) + \int_0^t \frac{\partial F_g}{\partial t}(s,X_{s-})\mathbb{I}_{\{g_{s-}=s\}}\mathrm{d}s + \int_0^t \frac{\partial F}{\partial t}(g_{s-},s,X_{s-})\mathbb{I}_{\{g_{s-}<s\}}\mathrm{d}s \\
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& \qquad \times \mathbb{I}_{\{X_{s-}+y>0\}}\mathbb{I}_{\{g_{s-}<s\}} N(\mathrm{d}s\times\mathrm{d}y) \\
& + \int_{[0,t]}\int_{(-\infty,0)} \left[F(s,s,X_{s-}+y) - F(g_{s-},s,X_{s-}) - y\frac{\partial F}{\partial x}(g_{s-},s,X_{s-})\right] \\
& \qquad \times \mathbb{I}_{\{X_{s-}+y\le 0\}}\mathbb{I}_{\{g_{s-}<s\}} N(\mathrm{d}s\times\mathrm{d}y),
\end{aligned}
$$

where $F_g(t,x) := F(t,t,x)$ for $t \geq 0$ and $x \leq 0$.

And now what?

Using Itô formula we can deduce, among others

- The infinitesimal generator of (g_t, t, X_t).

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  in terms of scale functions of X.
- Joint Laplace transform $\left(U_{\mathbf{e}_q}, X_{\mathbf{e}_q}\right)$.
- Solve optimal stopping problems related to corporate bankruptcy
- Resolve various aspects needed to solve

$$\inf_\tau \mathbb{E}[|\tau - g|^p].$$

Ingredients of the proofs

- Perturbed Lévy process. Revuz Yor (1999), Dassios Wu (2011).

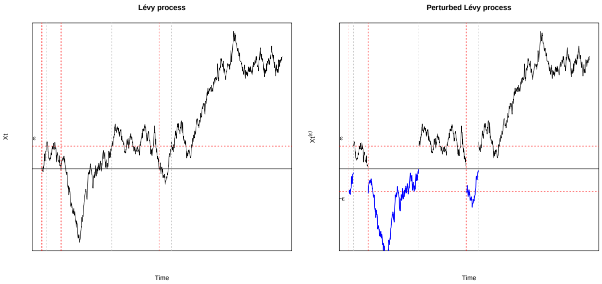

Figure: Left: Sample path of X. Right: Sample path of the perturbed process

- Use appropriate version of known Itô formula (e.g. Peskir's),properties of g_t, limiting arguments and local time.

Thank you