Rank-based models with listings and delistings

Theory and calibration

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LSE Statistics Research Showcase

Joint work in progress with Martin Larsson, Licheng Zhang (CMU)

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Long-term modelling of financial markets

THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

- Many market participants are interested in long-time horizons
  - Pension funds, Trust funds, Endowments, etc.
- We will focus on equity markets (stocks).
- Modelling any noisy system over a long period of time is challenging.

  → but may be rewarding!

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Questions:

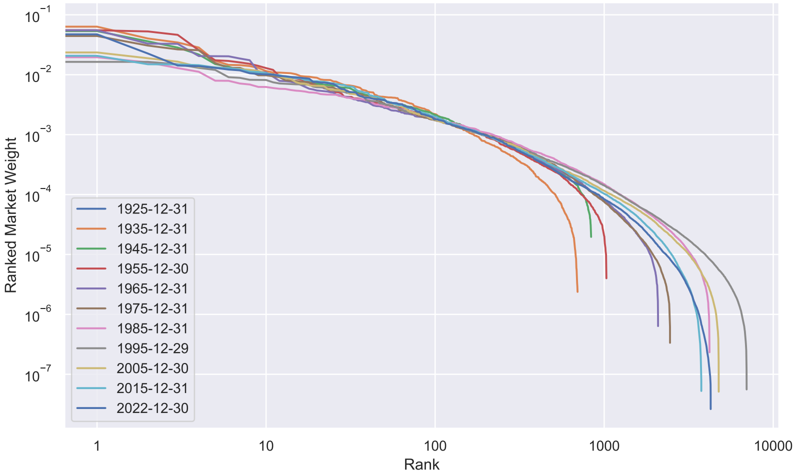
1. What features of equity markets persist over long-time horizons
2. Can we develop models capturing such features and procedures for statistical calibration?

Capital distribution curve

- Let $S_1, \dots, S_N$ denote the market capitalization processes of N companies.
- Set
$$\mu_i(t) = \frac{S_i(t)}{S_1(t) + \cdots + S_N(t)} \qquad i = 1, \dots, N$$
to be the market weights.
- The ranked market weights $\mu_{(1)} \geq \mu_{(2)} \geq \cdots \geq \mu_{(N)}$ are remarkably stable over time.

Capital distribution curve

Ranked market weights over time



Data Source: CRSP

First-order ranked based models

- Curve stability was first observed by Robert Fernholz who developed Stochastic Portfolio Theory (SPT) in 2002
  - SPT further developed together with Ioannis Karatzas, his students including Kardaras, Ruf and many others.

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- Letting $X_i = \log S_i$ the model postulates dynamics

$$dX_i(t) = \gamma_{r_i(t)}dt + \sigma_{r_i(t)}dW_i(t)$$

where $\gamma_i \in \mathbb{R}$, $\sigma_i > 0$, $r_i(t)$ is the rank asset i occupies at time t and W is an N-dimensional Brownian Motion.

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  where $\gamma_i \in \mathbb{R}$, $\sigma_i > 0$, $r_i(t)$ is the rank asset i occupies at time t and W is an N-dimensional Brownian Motion.
- Under the stability condition

$$\bar{\gamma} := \frac{1}{N}\sum_{k=1}^{N}\gamma_k > \frac{1}{n}\sum_{k=1}^{n}\gamma_k, \quad n = 1, \dots, N-1,$$

  the market weights are ergodic representing stability of the curve.
- When $\gamma_N \gg \gamma_i$ for every other i we say the model is Atlas-like.

Parameter calibration

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- Ranked log-caps: $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(N)}$.
- Ranked dynamics:

$$dX_{(k)}(t) = \gamma_k dt + \sigma_k d\widetilde{W}_k(t) + \frac{1}{2} d\Lambda_k(t) - \frac{1}{2} d\Lambda_{k-1}(t),$$

  where Λ_k is the local time at zero of $X_{(k)} - X_{(k+1)}$ and $\Lambda_0 = \Lambda_N = 0$.
- This term Λ_k activates when two particles collide $X_{(k)} = X_{(k+1)}$ and ensures the ordering $X_{(k)} \geq X_{(k+1)}$ persists

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- Volatility calibration: $\sigma_k^2 = \frac{d[X_{(k)}](t)}{dt}$,
  - → standard estimation using quadratic variation.
- Drift calibration: $\gamma_k - \bar{\gamma} = \frac{1}{2}\lim_{T\to\infty}(\frac{1}{T}\Lambda_{k-1}(T) - \frac{1}{T}\Lambda_k(T))$.
  - → Requires estimating collision rates $\lambda_k = \frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\Lambda_k(T)$.
  - → Efficient method using a so-called "Master formula" for portfolio generation available and developed in Fernholz (2002)

Calibrated Parameters

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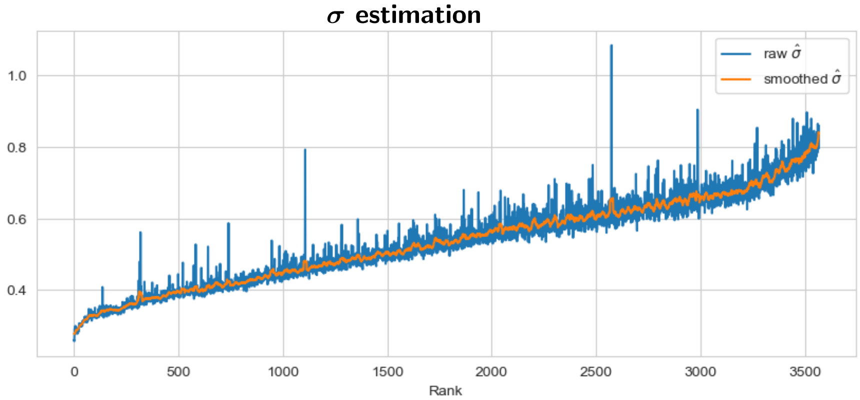
We use 51 years of CRSP data: 1973-2024 with $N = 3565$



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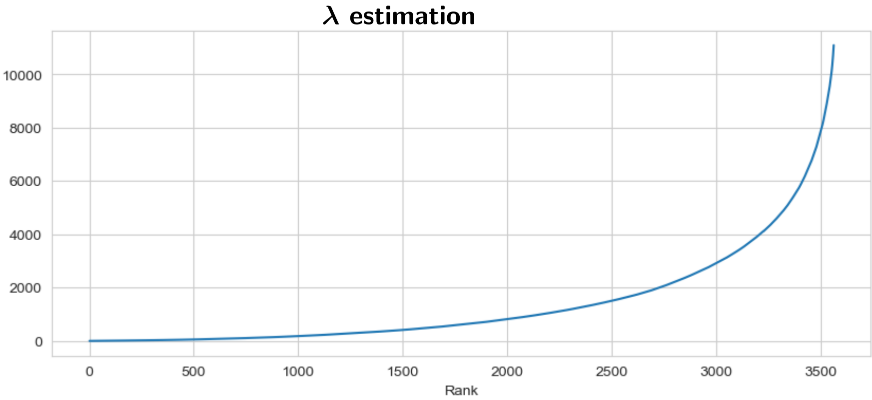

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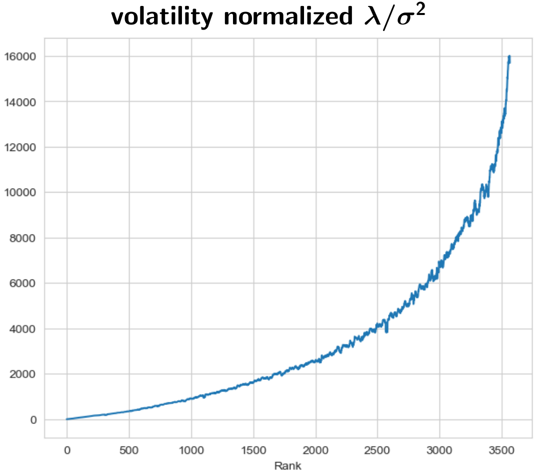

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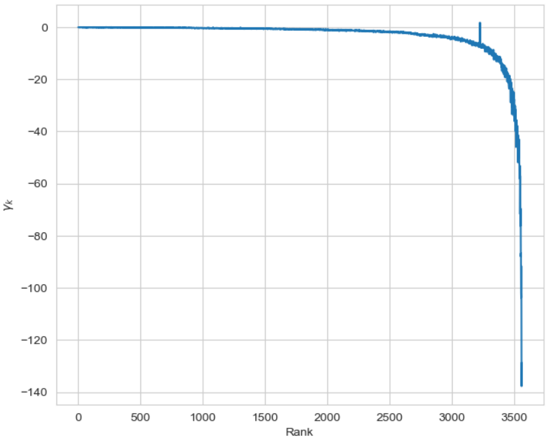
γ estimation



Not plotted: $\gamma_d \approx 11000$

Puzzles

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- Why are λ's increasing more and more rapidly?

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  - Directly linked to one-sided collision at final rank.
  - Artefact of the model due to fixed number of stocks.
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Issues seem to be linked with the smallest stock/fixed universe.

- We try to address this by allowing for listings and delistings.
  - How prevalent are they?
- Campbell & Wong (2024) identified these as important drivers for capital distribution curve stability.

Model Setup

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- We model a variable equity universe with a focus on developing estimators for empirical calibration,
  - Some recent literature on equity models with variable assets: Sarantsev & Karatzas (2016), Bayraktar, Kim & Tilva (2024)
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$$dX_i(t) = \gamma_{r_i(t)} dt + \sigma_{r_i(t)} dW_i(t), \quad \beta_i \leq t \leq \delta_i,$$

where $r_i(t)$ is the rank of asset i at time t among the listed assets.

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  where $r_i(t)$ is the rank of asset i at time t among the listed assets.
- We set $I(t) = \{i : t \in [\beta_i, \delta_i]\}$ and $N(t) = |I(t)|$.
- We assume nondegeneracy: $N(t) \geq 1$ and finite activity: $\sum_{t \leq T} \Delta N(t) < \infty$ for every $T > 0$.

Ranked Dynamics

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- Despite X_i being continuous on its lifetime $X_{()}$ experiences jumps at listing and delisting times.
- Moreover, the market capitalization $\Sigma(t) = \sum_{i \in I(t)} S_i(t)$, the market weights

$$\mu_i(t) = \frac{S_i(t)}{\Sigma(t)} 1_{\{i \in I(t)\}},$$

and the ranked market weights $\mu_{()}(t)$ experience jumps as well.

Illustration

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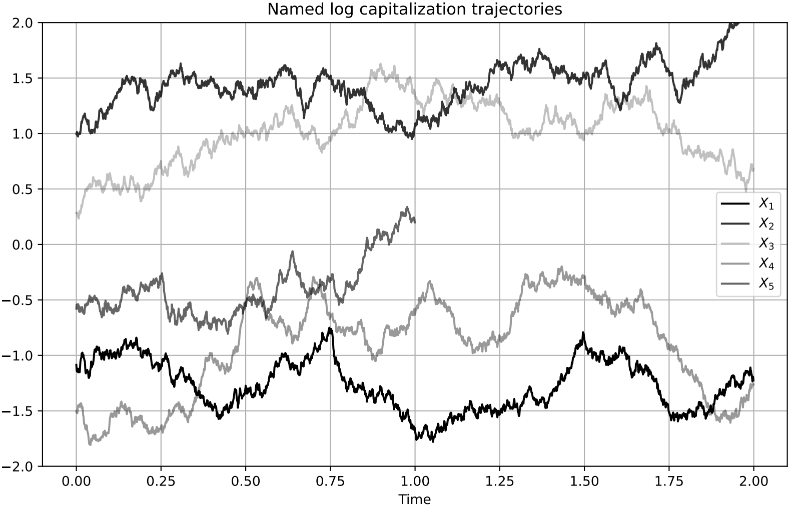

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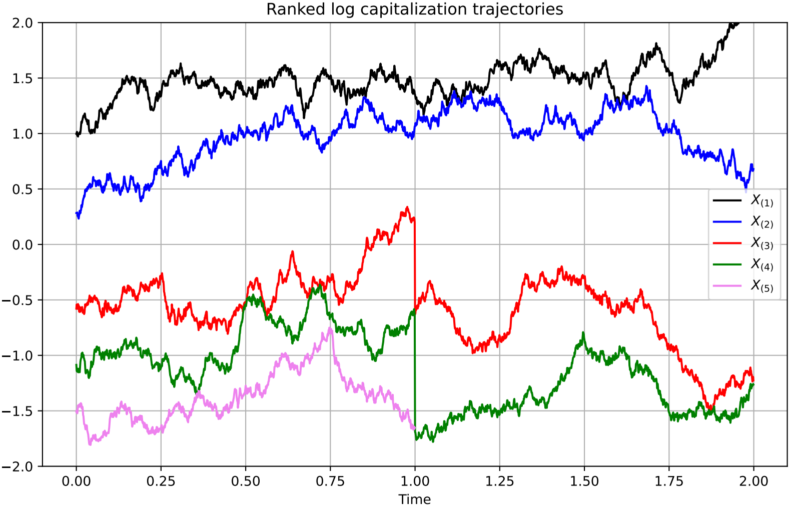

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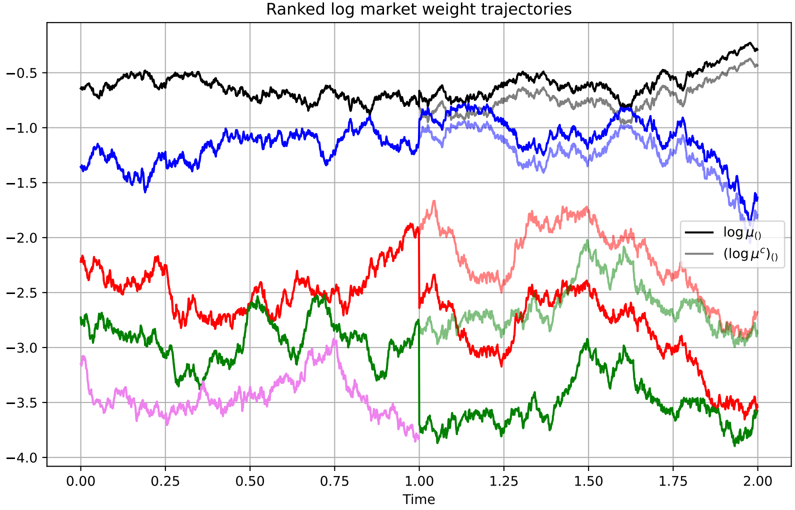

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- Jumps have a larger effect on the smaller ranks.
- Care is needed dealing with portfolios and the collision estimator.
- Indeed, the wealth $V^{\mathcal{M}}$ of an investor trading the market portfolio now satisfies

$$\log V^{\mathcal{M}}(t) = \log \Sigma^c(t),$$

  whereas classically $\log V^{\mathcal{M}}(t) = \log \Sigma(t)$.

Collision estimation with listings/delistings

- We developed a new "Master formula" for portfolio generation in this setting with listings and delistings.
- As in the classical setting the collision estimator can be derived by looking at the large-cap portfolio investing in the top k assets.
- In the fixed investment universe

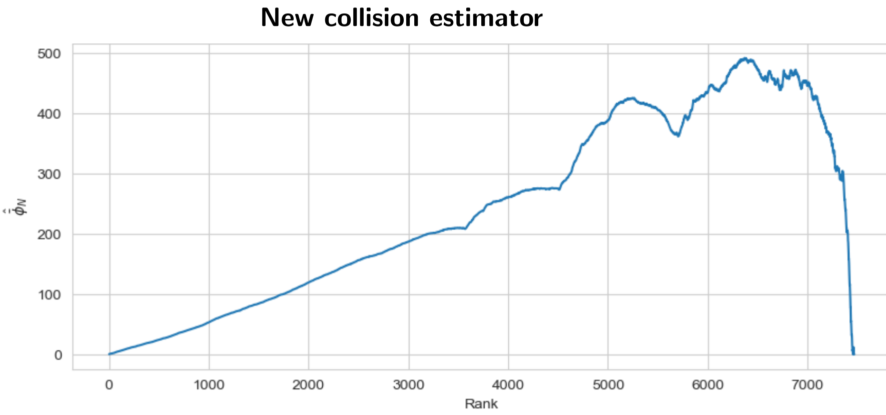
$$d\Lambda_k(t) = \frac{\mu_{(1)}(t) + \cdots + \mu_{(k)}(t)}{\mu_{(k)}(t)} d \log \left(\frac{V^{\mathcal{M}_k}}{V^{\mathcal{M}}} \times \frac{1}{\mu_{(1)} + \cdots + \mu_{(k)}} \right)(t)$$

- With listings/delistings (on the set $\{|I(t)| \geq k\}$):

$$d\Lambda_k(t) = \frac{\tilde{\mu}_{(1)}(t) + \cdots + \tilde{\mu}_{(k)}(t)}{\tilde{\mu}_{(k)}(t)} d \log \left(\frac{V^{\mathcal{M}_k}}{V^{\mathcal{M}}} \times \frac{1}{\tilde{\mu}_{(1)} + \cdots + \tilde{\mu}_{(k)}} \right)(t),$$

where $\log \tilde{\mu}(t) = \log \mu^c(t)$.

Collision estimation results



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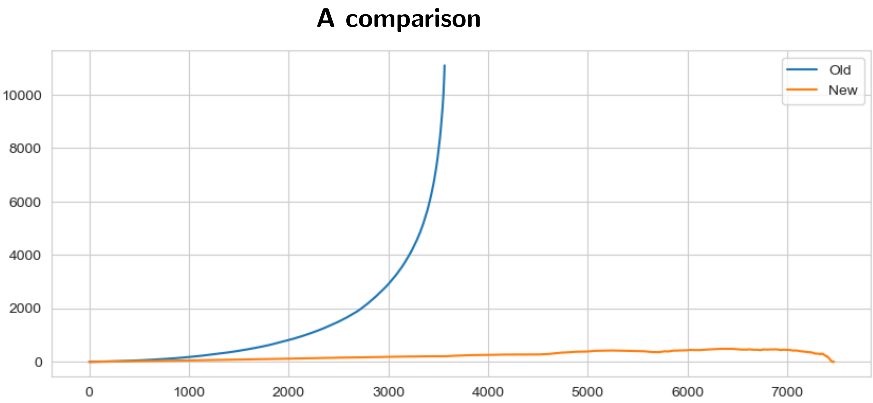

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where $\log\tilde{\mu}(t) = \log\mu^c(t)$.

The original estimator registers rank change caused by a listing/delisting as a collision.

- Although the original estimator is consistent in the fixed asset model, when applied to real data, it produces bias.
- The bias propogates effecting the smallest stocks the most since a listing/delisting at rank k causes a jump for each $\mu_{(\ell)}$, $\ell > k$.

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Particle Density

- What is a driver of collisions?
- Naively, we expect
  - More collisions, if highly volatile,
  - Fewer collisions if neighbours are positively correlated,
  - More collisions if particles tightly packed.

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- We define average particle density at rank k as

$$\phi_k = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{2n-1}{X_{(k-n)}(t) - X_{(k+n)}(t)} dt.$$

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What is the relationship between σ_k, ρ_k, ϕ_k and how is this related to the collision rates λ_k?

Local Model

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  with $d[W_i, W_j](t) = \rho dt$ for all $i \neq j$.
- This is an Ornstein–Uhlenbeck process with stationary measure $N(0, \Sigma_N)$ where $\Sigma_N = (1-\rho)I_N + \rho 1_{N \times N}$.

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  - The analysis leads to the same conclusion for any fixed quantile.
- We are interested in understanding the relationship between σ, ρ, λ and ϕ as $N \to \infty$.

Some order statistics results

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- We can write $X_i^N = \sqrt{\rho}Y + \sqrt{1-\rho}Z_i^N$ where Y and $(Z_i^N; i = -N, \ldots, N)$ are IID N(0,1) random variables.
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Proposition (I., Larsson, Zhang (2025+))

1. *Then density f^N on $[0,\infty)$ of the random variable $N(Z_{(0)}^N - Z_{(1)}^N)$ satisfies $\lim_{N\to\infty} f^N(0) = \sqrt{2/\pi}$.*
2. *The random variable $N(Z_{(-n)} - Z_{(n)})$ converges in distribution as $N \to \infty$ to a $\Gamma(2n, \sqrt{2/\pi})$ random variable.*
3. *The weak convergence in part two extends to the unbounded function $h(x) = 1/x$.*

Collision parameter in the local model

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- We have

$$\frac{\lambda_0^N}{N} = \lim_{\epsilon \to 0} \mathbb{E}\left[\frac{1}{T}\int_0^T \frac{1}{N\epsilon} 1_{\{X_{(0)}^N(t) - X_{(1)}^N(t) < \epsilon\}} d[X_{(0)}^N - X_{(1)}^N](t)\right]$$

$$= 2\sigma^2\sqrt{1-\rho} f^N(0) \longrightarrow 2\sigma^2\sqrt{1-\rho} \times \sqrt{\frac{2}{\pi}},$$

Collision parameter in the local model

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- and

$$\frac{\phi_0^N}{N} = \mathbb{E}\left[\frac{2n-1}{N(X_{(-n)} - X_{(n)})}\right] \longrightarrow (2n-1)\int_0^\infty \frac{\lambda^{2n}}{\Gamma(2n)} x^{2n-2} e^{-\sqrt{2/\pi}x} dx$$

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Collision parameter in the local model

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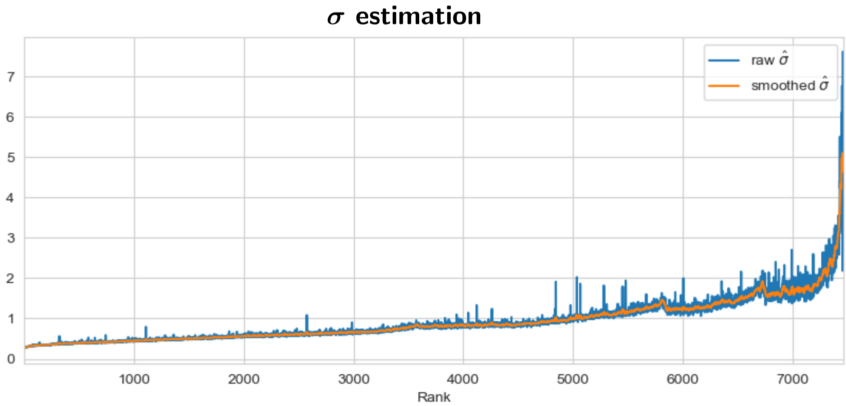
$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{\sqrt{1-\rho}}.$$

- Hence, $\phi_0^N \approx \frac{\lambda_0^N}{2\sigma^2(1-\rho)}$.
- Let see how the listing and delisting model performs with real data.

Collisions and Particle Density: Empirics



Collisions and Particle Density: Empirics

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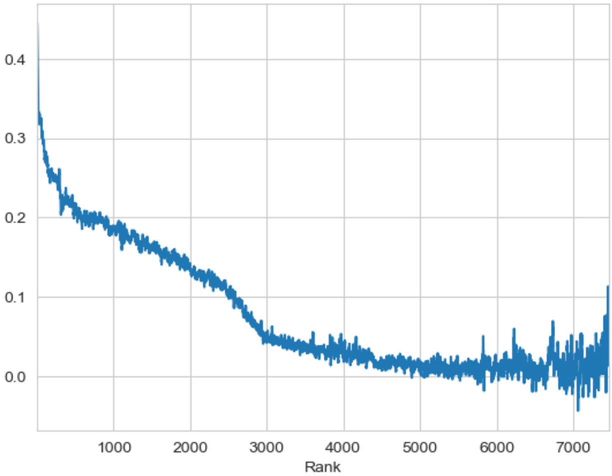
ρ-estimation



Collisions and Particle Density: Empirics

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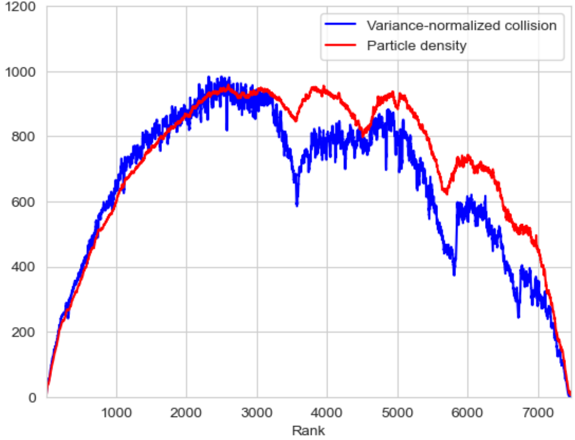
ϕ_k **vs** $\frac{\lambda_k}{2\sigma_k^2(1-\rho_k)}$ **with** $n = 5$.



A more granular look?

Conclusion and future work

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Summary of results

- Introduced a rank-based model with listings and delistings,
- Derived a new master formula for portfolio generation
- Derived collision estimator accounting for listings/delistings, which corrects bias of previously used estimator when applied to real data,
- Studied local model and connected collisions to particle density.

To be done:

- Estimate listing and delisting rates,
- Pick a (Markovian) birth/death mechanism for global model,
- Conduct numerical and simulation experiments for global model.
- Theoretical analysis of global model?

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Thank you!

Master Formula for Functional Generation

- In this setting we say a portfolio π is functionally generated if

$$\log\left(\frac{V^{\pi}(T)}{V^{\mathcal{M}}(T)}\right) = \log\left(\frac{G(\tilde{\mu}(T))}{G(\tilde{\mu}(0))}\right) + \Gamma(T)$$

for some function $G : \bigcup_d \mathbb{R}^d \to \mathbb{R}$ and process of finite variation Γ.

- Here $\log \tilde{\mu}(t) = \log \mu^c(t)$ $\longrightarrow$ differs from the standard setting.

Theorem (I., Larsson, Zhang 2025+)

For a C^2 function F, the portfolio $\pi_i(t) = \sum_k \eta_k(t-) 1_{\{r_i(t-)=k\}}$ which invests

$$\eta_k(t) = \tilde{\mu}_{(k)}(t)\left(\partial_k \log F(\tilde{\mu}_{()}(t)) + \frac{1 - \sum_{\ell=1}^{N(t)} \tilde{\mu}_{(\ell)}(t)\partial_\ell \log F(\tilde{\mu}_{()}(t))}{\sum_{\ell=1}^{N(t)} \tilde{\mu}_{(\ell)}(t)}\right) 1_{\{k \le |I(t)|\}}$$

in the asset at rank k is functionally generated by $G(x) = F(x_{()})$ with

$$d\Gamma(t) = -\frac{1}{2}\sum_{k,\ell=1}^{N(t)} \frac{\partial_{k\ell} F(\tilde{\mu}(t))}{F(\tilde{\mu}(t))} d[\tilde{\mu}_{(k)}, \tilde{\mu}_{(\ell)}](t) - \frac{1}{2}\sum_{k=1}^{N(t)} (\eta_k(t) - \eta_{k+1}(t)) d\Lambda_k(t).$$

Collision rates estimation

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- Applying this with the function $F(x) = x_{(1)} + \cdots + x_{(k \wedge d)}$ for $x \in \mathbb{R}^d$ yields that the large-cap portfolio of size k,

$$\pi_i(t) = \frac{\tilde{\mu}_i(t)}{\tilde{\mu}_{(1)}(t) + \cdots + \tilde{\mu}_{(k \wedge N(t))}(t)} 1_{\{r_i(t-) \leq k\}}$$
$$= \frac{\mu_i(t-)}{\mu_{(1)}(t-) + \cdots + \mu_{(k \wedge N(t-))}(t-)} 1_{\{r_i(t-) \leq k\}}$$

has wealth process

$$\log\left(\frac{V^{\mathcal{M}_k}(T)}{V^{\mathcal{M}}(T)}\right) = \log\left(\frac{\tilde{\mu}_{(1)}(T) + \cdots + \tilde{\mu}_{(k \wedge N(T))}(T)}{\tilde{\mu}_{(1)}(0) + \cdots + \tilde{\mu}_{(k \wedge N(0))}(0)}\right) + \frac{\tilde{\mu}_{(k \wedge N(t))}(t)}{\tilde{\mu}_{(1)}(t) + \cdots + \tilde{\mu}_{(k \wedge N(t))}(t)} d\Lambda_k(t),$$

with the convention that $d\Lambda_k(t) = 0$ on $\{|I(t)| < k\}$.

Discretized Estimators

- Old estimator for local time

$$\frac{1}{T}\sum_{i=0}^{M-1}\frac{S_{(1)}(t_i)+\cdots+S_{(k)}(t_i)}{S_{(k)}(t_i)}\log\left(\frac{S_{(1)}(t_i+1)+\cdots+S_{(k)}(t_i+1)}{S_{n_1(t_i)}(t_i+1)+\cdots+S_{n_k(t_i)}(t_i+1)}\right),$$

where $n_k(t)$ is the name occupying the k'th rank at time t.

- New estimator for local time:

$$\frac{1}{Tp(k)}\sum_{i=0}^{M-1}1_{\{|I(t_i)|\geq k\}}\frac{S_{(1)}(t_i)+\cdots+S_{(k)}(t_i)}{S_{(k)}(t_i)}$$
$$\times\log\left(\frac{S_{J_{(1)}^{t_i,t_i+1}}(t_i+1)+\cdots+S_{J_{(k)}^{t_i,t_i+1}}(t_i+1)}{S_{J_{(1)}^{t_i,t_i}}(t_i+1)+\cdots+S_{J_{(k)}^{t_i,t_i}}(t_i+1)}\right),$$

where

- $p(k)=\frac{1}{M}\sum_{i=0}^{M-1}1\{|I(t_i)>k|\}$,
- $J_{(\ell)}^{t,s}$ is the name of of ℓ'th largest market cap based on time s values and out of only the names that are listed at time t and $t+1$.