Mixed Impulse/Stopping Nonzero-Sum Stochastic Differential Games

Davide De Santis (joint work with Luciano Campi)

Department of Statistics, London School of Economics

Introduction

We consider a nonzero-sum stochastic game between two players, Player 1 and Player 2 where Player 1 uses impulse controls while Player 2 can stop the game any time he wants. Both want to maximize given objective functionals. Mathematically speaking we are given:

- a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with $(\mathcal{F}_t)_{t\geq 0}$, the filtration generated by the standard Brownian motion $(W_t)_{t\geq 0}$;
- an uncontrolled state variable $X \equiv X^x$ whose dynamics is
$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{1}$$
where the coefficients satisfy all is needed to grant existence of a unique strong solution (e.g. Lipschitz continuity).
- Player 1 can affect X's dynamics with impulse controls $u = (\tau_n, \delta_n)_{n\geq 1}$ where τ_n is an increasing sequence of stopping times such that $\tau_n \uparrow \infty$ as $n \uparrow \infty$, while δ_n is the size of the corresponding impulse, i.e. $\delta_n \in L^0(\mathcal{F}_{\tau_n})$. Any impulse δ_n brings the state variable from $X_{\tau_n -}$ to its new value $X_{\tau_n} = X_{\tau_n^-} + \delta_n$. The controlled state variable will be denoted by $X^{x,u}$. Moreover, every time Player 1 intervenes he faces a cost, say $\phi(X_{\tau_n -}, \delta_n)$, which may depend on the state variable before the intervention as well as on the impulse.
- Player 2 can stop the game by choosing any stopping time η with values in $[0, \infty]$. Player 2 can get something, say $\psi(X_{\tau_n -}, \delta_n)$, any time Player 1 intervenes.
- Both players wants to maximize their respective objectives, which are given by

$$J_1(x; u, \eta) = \mathbb{E}\left[\int_0^\eta e^{-r_1 t} f(X_t^{x,u})dt - \sum_{n:\tau_n \leq \eta} e^{-r_1 \tau_n}\phi(X_{\tau_n -}, \delta_n) + e^{-r_1 \sigma} h(X_\eta)\mathbf{1}_{(\eta<\infty)}\right] \tag{2}$$
$$J_2(x; u, \eta) = \mathbb{E}\left[\int_0^\eta e^{-r_2 t} g(X_t^{x,u})dt + \sum_{n:\tau_n \leq \eta} e^{-r_2 \tau_n}\psi(X_{\tau_n -}, \delta_n) + e^{-r_2 \sigma} k(X_\eta)\mathbf{1}_{(\eta<\infty)}\right] \tag{3}$$

Game Setting

First, we need to introduce players' strategies, defined as follows:

- Player 1's strategy is $u = (\tau_n, \delta_n)_{n\geq 1}$, where $(\tau_n)_{n\geq 1}$ is a sequence of stopping times such that $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_n \uparrow \infty$ and $\delta_n \in L^0(\mathcal{F}_{\tau_n^-})$ with $\delta_n : \mathbb{R}^d \to Z,\ Z \subseteq \mathbb{R}^d$.
- Player 2's strategy is a stopping time $\eta \in \mathcal{T},\ \eta : \Omega \to [0, \infty]$.

Hence, the controlled process is:
$$X_t^{x;u} := x + \int_0^t b(X_s^{x;u})ds + \int_0^t \sigma(X_s^{x;u})dW_s + \sum_{n:\tau_n \leq t} \delta_n$$
We denote by $\mathcal{A}_x$ the set of the x-admissible pairs.

Nash Equilibrium: *Given $x \in \mathbb{R}^d$, we say that $(u, \eta) \in \mathcal{A}_x$ is a Nash equilibrium of the game if*
$$J_1(x; u^*, \eta^*) \geq J_1(x; u, \eta^*)\ \forall\ u \text{ s.t. } (u, \eta^*) \in \mathcal{A}_x$$
$$J_2(y; u^*, \eta^*) \geq J_2(x; u^*, \eta)\ \forall\ \eta \text{ s.t. } (u^*, \eta) \in \mathcal{A}_x$$
Finally, the equilibrium payoffs of the game are defined as follows: if $x \in \mathbb{R}^d$ and a Nash equilibrium $(u^*, \eta^*) \in \mathcal{A}_x$ exists, we set for $i \in \{1, 2\}$
$$V_i(x) := J_i(x; u^*, \eta^*)$$

The Quasi-Variational Inequalities System

The differential problem that should be satisfied by the equilibrium payoff functions of our game is as follows:
Assume that V_1 and V_2 are defined for each $x \in \mathbb{R}^d$ and that there exists at least a function δ from $\mathbb{R}^d$ to Z such that:
$$\{\delta(x)\} = \text{argmax}_{\delta\in Z}\{V_1(x + \delta) - \phi(x, \delta)\} \tag{4}$$
for each $x \in \mathbb{R}^d$. We define the following two intervention operators:
$$\mathcal{M}V_1(x) = V_1(x + \delta(x)) - \phi(x, \delta(x)) \tag{5}$$
$$\mathcal{H}V_2(x) = V_2(x + \delta(x)) + \psi(x, \delta(x)) \tag{6}$$
for each $x \in \mathbb{R}^d$. Moreover, we assume $V_1, V_2 \in C^2(\mathbb{R}^d)$ and define
$$\mathcal{A}V = bV_x + \frac{1}{2}\sigma^2 V_{xx}$$
We are interested in the following quasi-variational inequalities for V_1, V_2:
$$\begin{array}{lr} \mathcal{M}V_1 - V_1 \leq 0 & \text{everywhere} \quad (7)\\ V_2 - k \geq 0 & \text{everywhere} \quad (8)\\ \mathcal{H}V_2 - V_2 = 0 & \text{in } \{\mathcal{M}V_1 - V_1 = 0\} \quad (9)\\ V_1 = h & \text{in } \{V_2 = k\} \quad (10)\\ \max\{\mathcal{A}V_1 - r_1V_1 + f, \mathcal{M}V_1 - V_1\} = 0 & \text{in } \{V_2 > k\} \quad (11)\\ \max\{\mathcal{A}V_2 - r_2V_2 + g, k - V_2\} = 0 & \text{in } \{\mathcal{M}V_1 - V_1 < 0\} \quad (12) \end{array}$$
The intuition behind these conditions is as follows:

- (7): means that is not always optimal to intervene and is a standard condition in impulse control theory [4], [5];
- (8): if Player 2 plays $\eta = 0$ he gains $k(x)$, since this is a suboptimal strategy we have $V_2 \geq k\ \forall\, x \in \mathbb{R}^d$;
- (9): by definition of Nash equilibrium we expect that Player 2 does not lose anything when Player 1 intervenes [1];
- (11): before Player 2 stops Player 1 plays a classic one-player impulse game;
- (12): when Player 1 does not intervene Player 2 solves his own optimal stopping problem.

The Verification theorem

Theorem : *Let V_1, V_2 be functions from $\mathbb{R}^d$ to $\mathbb{R}$. Assume that (4) holds and set $\mathcal{C}_1 := \{\mathcal{M}V_1 - V_1 < 0\}$ and $\mathcal{C}_2 := \{V_2 - k > 0\}$ with $\mathcal{M}V_1$ as in (5). Moreover, assume that:*

- *V_1 and V_2 are solutions of the system of QVIs;*
- *$V_1 \in C^2(\mathcal{C}_2 \setminus \partial\mathcal{C}_1) \cap C^1(\mathcal{C}_2) \cap C(\mathbb{R}^d)$, $V_2 \in C^2(\mathcal{C}_1 \setminus \partial\mathcal{C}_2) \cap C^1(\mathcal{C}_1) \cap C(\mathbb{R}^d)$, and both of them have polynomial growth;*
- *$\partial\mathcal{C}_i$ is a Lipshitz surface[1], and V_i's second order derivatives are locally bounded near $\partial\mathcal{C}_i$.*

Finally, let $x \in \mathbb{R}^d$ and assume that $(u^*, \eta^*) \in \mathcal{A}_x$, with $u^* = (\tau_n, \delta_n)_{n\geq 1}$ such that $\tau_n = \inf\{t > \tau_{n-1}; X_t \in \partial\mathcal{C}_1\}$ and $\{\delta_n\} = \text{argmax}_{\delta\in Z}\{V_1(X_{\tau_n^-} + \delta) - \phi(X_{\tau_n^-}, \delta)\}$, and $\eta^* = \inf\{t \geq 0 : V_2(X_t) = k(X_t)\}$.
Then, (u^*, η^*) is a Nash Equilibrium and $V_i = J_i(x; u^*, \eta^*)$ for $i \in \{1, 2\}$.

Remark: We have proved an alternative version of the above theorem providing an analogous result. The difference relies in changing the assumptions in order to use Corollary 4 in [7] instead of using the approximation arguments in [8] to be able to apply Itô's Formula during the proof.

Work in Progress

We are currently

- studying some examples in which Player 1, the controller, either induce Player 2, the stopper, to stop or he does not;
- Looking for economically/financially relevant examples. In particular, it might be interesting the case in which both the controller and the stopper are looking for high (or low) values of the underlying process. This situation might resemble a manager (controller) vs investor (stopper) kind of interaction, with the manager controlling the production in an attempt to maximize firm's profits while the investor can decide to shut it down, stopping the game, in case its outcome is not satisfactory.

Forthcoming Research

Going beyond the verification theorem, which requires too much regularity, we want to solve the game in a viscosity setting. In particular, we believe that, thanks to viscosity solution [6, 2] and stochastic Perron's method [3], the use of numerical methods directly on the system of quasi-variational inequalities should be possible.

References

[1] René Aïd, Matteo Basei, Giorgia Callegaro, Luciano Campi, and Tiziano Vargiolu. Nonzero-sum stochastic differential games with impulse controls: a verification theorem with applications. *arXiv preprint arXiv:1605.00039*, 2018.

[2] Parsiad Azimzadeh. A zero-sum stochastic differential game with impulses, precommitment, and unrestricted cost functions. *Applied Mathematics & Optimization*, pages 1–32, 2017.

[3] E. Bayraktar and M. Srbu. Stochastic perron's method and verification without smoothness using viscosity comparison: The linear case. *Proceedings of the American Mathematical Society*, 140(10):3645–3654, 2012.

[4] Giulio Bertola, Wolfgang Runggaldier, and Kazuhiro Yasuda. On classical and restricted impulse stochastic control for the exchange rate. *Applied Mathematics and Optimization*, 74(2):423–454, October 2016.

[5] Abel Cadenillas and Fernando Zapatero. Optimal central bank intervention in the foreign exchange market. *Journal of Economic Theory*, 87(1):218–242, July 1999.

[6] Andrea Cosso. Stochastic differential games involving impulse controls and double-obstacle quasi-variational inequalities. *SIAM Journal on Control and Optimization*, 51(3):2102–2131, 2013.

[7] Nikolaj Vladimirovič Krylov. *Controlled diffusion processes*, volume 14. Springer Science & Business Media, 2008.

[8] Bernt Karsten Øksendal and Agnes Sulem. *Applied stochastic control of jump diffusions*, volume 498. Springer, 2005.

[1] I.e. it is locally the graph of a Lipschitz function.