Predicting the last zero of a spectrally negative Lévy process

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Abstract

Given a spectrally negative Lévy process drifting to infinity, we consider the last time g the process is below zero. We are interested in finding a stopping time which is as close as possible to g. In the L_1 setting, we show that an optimal stopping time is given by a first passage time above a level based on the convolution with itself of the distribution function of minus the overall supremum of the process. The proof is based on a direct approach without the need to make use of stochastic calculus.

For some more general metrics the problem is more challenging and can be transformed into an optimal stopping problem for a three-dimensional Markov process involving the last passage time. We show that the solution of this optimal stopping problem is given by the first time that the Lévy process crosses a non-increasing, non-negative curve which depends on the time spent above zero.

Introduction

In recent years last exit times have been studied in several areas of applied probability, e.g. risk theory or degradation models. In the risk theory setting we might consider the Cramér–Lundberg process, which is a process consisting of a deterministic drift plus a compound Poisson process which has only negative jumps which typically models the capital of an insurance company. A key quantity of interest is the time of ruin τ_0, i.e. the first time the process becomes negative. Suppose the insurance company has funds to endure negative capital for some time. Then another quantity of interest is the last time g that the process is below zero. In a more general setting we may consider a spectrally negative Lévy process (see for example Figure 1) instead of the classical risk process.

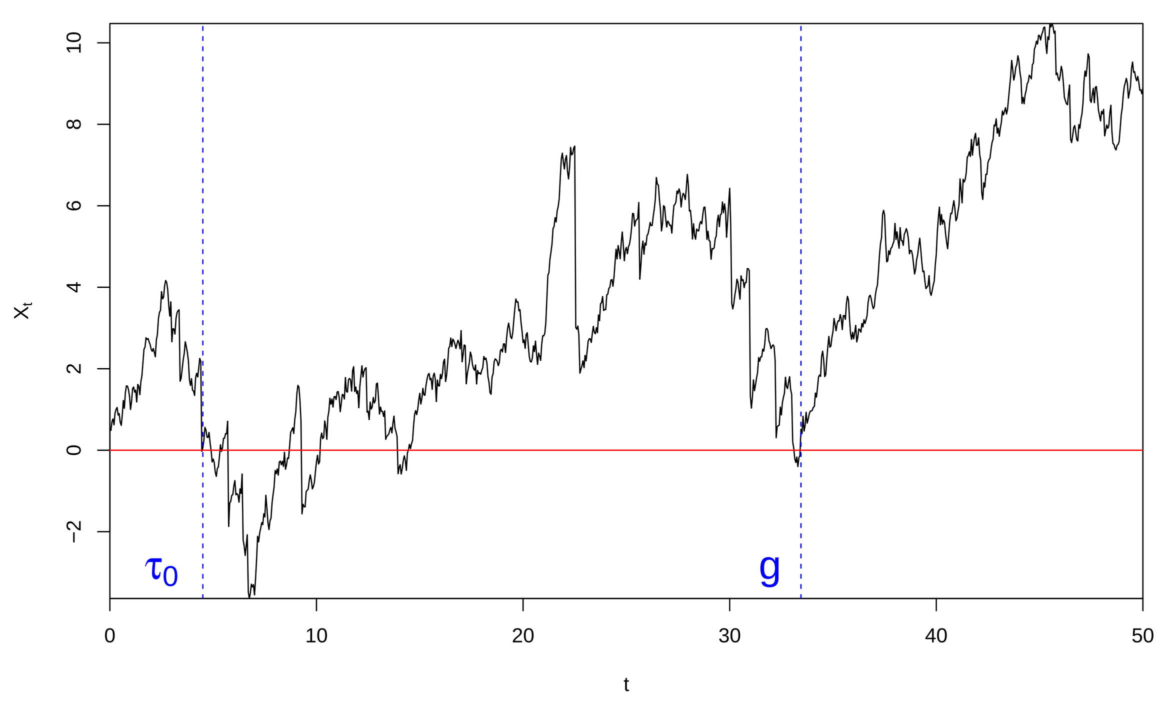

Figure 1: Cramér–Lundberg process with Brownian component.

Formulation of the problem

Let X be a spectrally negative Lévy process drifting to infinity, i.e. $\lim_{t\to\infty} X_t = \infty$, starting from 0. Suppose that X has Lévy triple (c, σ, Π) where $c \in \mathbb{R}$, $\sigma \geq 0$ and Π is the so-called Lévy measure concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)} (1 \wedge x^2)\Pi(dx) < \infty$. The characteristic exponent defined by $\Psi(\theta) := -\log(\mathbb{E}(e^{i\theta X_1}))$ takes the form

$$\Psi(\theta) = ic\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} (1 - e^{i\theta x} + i\theta x \mathbb{I}_{\{x>-1\}})\Pi(dx).$$

Let $W^{(q)}$ and $Z^{(q)}$ the scale functions corresponding to the process X. That is, $W^{(q)}$ is such that $W^{(q)}(x) = 0$ for $x < 0$, and is characterised on $[0, \infty)$ as a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q} \quad \text{for } \beta > \Phi(q) \quad \text{and} \quad Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y)dy$$

where ψ and Φ are, respectively, the Laplace exponent and its right inverse given by

$$\begin{aligned} \psi(\lambda) &= \log \mathbb{E}(e^{\lambda X_1}) & \lambda \geq 0 \\ \Phi(q) &= \sup\{\lambda \geq 0 : \psi(\lambda) = q\} & q \geq 0. \end{aligned}$$

In particular, $\boldsymbol{F(x) := \mathbb{P}(-\underline{X}_\infty \leq x) = \psi'(0+)W(x)}$ where $\boldsymbol{\underline{X}_\infty = \inf_{t\geq 0} X_t}$. Denote by g as the last passage time below 0, i.e.

$$g = \sup\{t \geq 0 : X_t \leq 0\}. \tag{1}$$

with the convention $\sup \emptyset = 0$. Clearly, up to any time $t \geq 0$ the value of g is unknown (unless X is trivial), and it is only with the realisation of the whole process that we know that the last passage time below 0 has occurred. However, this is often too late: typically one would like to know how close X is to g at any time $t \geq 0$ and then take some action based on this information. We search for a stopping time τ_* of X that is as "close" as possible to g. Consider the optimal prediction problem

$$V_* = \inf_{\tau\in\mathcal{T}} \mathbb{E}|g - \tau|, \tag{2}$$

where $\mathcal{T}$ is the set of all stopping times.

Optimal stopping problem

We state the equivalence between the optimal prediction problem and an optimal stopping problem mentioned earlier.

Lemma 1. *Consider the standard optimal stopping problem*

$$V = \inf_{\tau\in\mathcal{T}} \mathbb{E}\left(\int_0^\tau G(X_s)ds\right), \tag{3}$$

where the function G is given by $\boldsymbol{G(x) = 2\psi'(0+)W(x) - 1}$ for $x \in \mathbb{R}$. Then the stopping time which minimises (2) is the same which minimises (3). In particular $V_* = V + \mathbb{E}(g)$.

To find the solution of the optimal stopping problem (3) we will expand it to an optimal stopping problem for a strong Markov process X with starting value $X_0 = x \in \mathbb{R}$. Specifically, we define the function $V : \mathbb{R} \mapsto \mathbb{R}$ as

$$V(x) = \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\left(\int_0^\tau G(X_s)ds\right). \tag{4}$$

Thus $V_* = V(0) + \mathbb{E}(g)$.

Theorem 1. *Suppose that X is a spectrally negative Lévy process drifting to infinity with Lévy measure Π satisfying $\int_{(-\infty,-1)} \boldsymbol{x^2\Pi(dx) < \infty}$. Then there exists some $\boldsymbol{a^* \in [x_0, \infty)}$ such that an optimal stopping time in (4) is given by*

$$\tau^* = \inf\{t \geq 0 : V(X_t) = 0\} = \inf\{t \geq 0 : X_t \geq a^*\}.$$

The optimal stopping level a^* is defined by $\boldsymbol{a^* = \inf\{x \in \mathbb{R} : H(x) \geq 1/2\}}$, where H is the convolution of F with itself, i.e.,

$$H(x) = \int_{[0,x]} F(x - y)dF(y)$$

Furthermore, V is a non-decreasing, continuous function satisfying the following:

*i) If X is of **infinite variation** or **finite variation** with $\boldsymbol{F(0)^2 < 1/2}$. Hence, $a^* > 0$ is the median of the distribution function H, i.e. is the unique value which satisfies $\boldsymbol{H(a^*) = \frac{1}{2}}$. The value function is given by*

$$V(x) = \frac{2}{\psi'(0+)}\int_x^{a^*} H(y)dy - \frac{a^* - x}{\psi'(0+)}\mathbb{I}_{\{x\leq a^*\}} \tag{5}$$

*Moreover, there is **smooth fit** at a^* i.e. $V'(a^*-) = 0 = V'(a^*+)$.*

*ii) If X is of **finite variation** with $\boldsymbol{F(0)^2 \geq 1/2}$ then $\boldsymbol{a^* = 0}$ and $\boldsymbol{V(x) = \frac{x}{\psi'(0+)}\mathbb{I}_{\{x\leq 0\}}}$. In particular, there is **continuous fit** at $a^* = 0$ i.e. $V(0-) = 0$ and there is no smooth fit at a^* i.e. $V'(a^*-) > 0$.*

Generalisation to any metric d

Denote $\boldsymbol{g_t}$ as the last time that the process is strictly below zero before time t, i.e.,

$$g_t = \sup\{0 \leq s \leq t : X_s < 0\},$$

with the convention $\sup \emptyset = 0$. Consider a continuous **convex function** $d : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$. Denote as $h(x, y)$ the right derivative of d with respect to x for fixed $y \geq 0$, i.e. $h(x, y) := \partial/\partial x d_+(x, y)$. We want to solve the optimal prediction problem

$$V_* = \inf_{\tau\in\mathcal{T}} \mathbb{E}(d(\tau, g)) \tag{6}$$

The following lemma establishes a equivalence between the optimal prediction problem and an optimal stopping problem.

Lemma 2. *Let X be a spectrally negative Lévy process drifting to infinity and $d : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ a continuous convex function such that $\mathbb{E}(\boldsymbol{d(0, g)}) < \infty$. Then the optimal prediction problem (6) is equivalent to the optimal stopping problem*

$$V = \inf_{\tau\in\mathcal{T}} \mathbb{E}\left(\int_0^\tau G(g_s, s, X_s)ds\right), \tag{7}$$

where the function G is given by

$$G(\gamma, s, x) = h(s, \gamma)\psi'(0+)W(x) + \mathbb{E}_x(h(s, g + s)\mathbb{I}_{\{\tau_0^- < \infty\}}),$$

for $0 \leq \gamma \leq s$ and $x \in \mathbb{R}$. We have that $V_* = V + \mathbb{E}(d(0, g))$.

We expand the problem for the three dimensional strong Markov process $\{(g_t, t, X_t), t \geq 0\}$. For every $0 \leq \gamma \leq t$ and $x \in \mathbb{R}$ we define

$$V(\gamma, t, x) = \inf_{\tau\in\mathcal{T}} \mathbb{E}_{\gamma,t,x}\left[\int_0^\tau G(g_{t+s}, t + s, X_{t+s})ds\right].$$

Therefore we have that $V_* = V(0, 0, 0) + \mathbb{E}(d(0, g))$.

Example 1. *Consider a convex non-decreasing and continuous function $f : \mathbb{R}_+ \mapsto \mathbb{R}$ and define $\boldsymbol{d(x, y) = f(|x - y|)}$. Thus the function d is convex in $\mathbb{R}^2$ and*

$$G(\gamma, s, x) = f'_+(s - \gamma)\psi'(0+)W(x) - \mathbb{E}_x(f'_+(g)\mathbb{I}_{\{g>0\}})$$

In particular if we choose $\boldsymbol{f(x) = x^\alpha}$ for $\alpha \geq 1$ the function G reads as

$$G(\gamma, s, x) = \alpha(s - \gamma)^{\alpha-1}\psi'(0+)W(x) - \alpha\mathbb{E}_x(g^{\alpha-1}\mathbb{I}_{\{g>0\}}).$$

□

Under some assumptions we show that there exists a function $b^* : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma \mapsto b^*(\gamma, t)$ is non-decreasing and $t \mapsto b^*(\gamma, t)$ is non-increasing. It satisfies that the stopping time

$$\tau_{b^*} = \inf\{s \geq 0 : X_{s+t} \geq b^*(g_{s+t}, s + t)\}$$

is optimal for (7). Moreover, in the case $d(x, y) = |x - y|^n$ for $n \geq 2$ we have that $b^*(\gamma, t) = b(t - \gamma)$ where $b : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a non-increasing function such that $b(0) = \infty$. Hence

$$\tau_{b^*} = \inf\{s \geq 0 : X_{s+t} \geq b(s + t - g_{s+t})\}.$$

Notice that $\boldsymbol{t - g_t}$ represents the time spent above zero by the process X after the last visit to the interval $(-\infty, 0)$.