Bursting Bubbles in a Macroeconomic Model*

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Abstract

This paper identifies the conditions and mechanisms that give rise to stochastic bubbles that are expected to collapse. To illustrate the essence of the emergence of stochastic bubbles, we first present a toy model, and then we present a full-fledged macro-finance model of intangible capital and show that stochastic stock bubbles attached to intangible capital emerge in the process of spillover of technological innovation. We show that the dynamics with stochastic bubbles, which is characterized by unbalanced growth, is a temporary deviation from a balanced growth path in which asset prices equal the fundamentals.

Keywords: balanced growth, intangible capital, stochastic bubbles, technological innovation, unbalanced growth.

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1 Introduction

Asset price bubbles are situations where asset prices exceed the fundamental values defined by the expected discounted present value of dividends. Although asset

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price bubbles are commonly discussed in the popular press and there is some empirical support,[1] the dominant view of modern macro-finance theory is that bubbles are either not possible in rational equilibrium models or even if they are, a situation in which bubbles occur is a special circumstance and hence fragile. Indeed, as an influential paper by Santos and Woodford (1997, Theorem 3.3, Corollary 3.4) shows, there is a fundamental difficulty in generating asset bubbles in real assets that pay dividends, such as stocks, land, and housing. Because of Santos and Woodford (1997)'s "Bubble Impossibility Theorem", there seems to be a presupposition in the macro-finance literature that asset prices should reflect the fundamentals. In these circumstances, Hirano and Toda (2025) challenge the conventional view. Within workhorse macroeconomic models, including overlapping generations and infinite-horizon models of Bewley's types, they prove the existence of bubbles attached to real assets and establish the Bubble Necessity Theorem, i.e., under some conditions, the only possible equilibrium is one that features asset price bubbles with the non-negligible bubble sizes relative to the economy.

In this paper, we consider environments with aggregate risk and advance the findings of Hirano and Toda (2025). In particular, this paper examines stochastic bubbles that are expected to collapse. We identify the economic conditions and mechanisms that give rise to stochastic bubbles. To describe stochastic bubbles, we consider regime switching between two states, i.e., one (UG) characterized by "unbalanced growth" where different factors of production grow at different rates, and the other (BG) characterized by "balanced growth" where they grow at the same rate. Once the state of BG arises, the macroeconomy will remain in that state, i.e., the state characterized by balanced growth is an absorbing state. The source of aggregate risk arises from this regime change.

We will derive three main results. First, in Section 2, to illustrate the essence of the emergence of stochastic bubbles, we present a toy model of two-period overlapping generations endowment economies with two sectors, in which there are land and income generating sectors. In the land sector, land yields rents, which serves as a means of saving for the young. In the income-generating sector, each young is endowed with income (endowments) exogenously. The productivity growth rates of the two sectors may be different. The young's income takes two states, where in state UG, the productivity growth rate of the income-generating sector is higher, and hence the young's endowments grow faster than land rents, exhibiting unbalanced growth. On the other hand, in the state of BG, the productivity growth

[1]Kindleberger (2000, Appendix B) documents 38 bubbly episodes in the 1618–1998 period.

rates are equal between the two sectors, and hence the young's endowments and land rents grow at the same rate, exhibiting balanced growth. In this setting, we show that land price bubbles that are expected to collapse emerge as the unique equilibrium outcome in transitional dynamics with unbalanced growth, and once the state of the macroeconomy transitions to BG, land price bubbles collapse. After identifying the conditions for the emergence of stochastic bubbles, as robustness, we will also show that these results will hold even when we consider the case with multiple savings vehicles, in which case stochastic bubbles in the values of multiple assets simultaneously emerge.

Second, based on the insights obtained in Section 2, in Section 3, we consider stock price bubbles. As highlighted by Scheinkman (2014, p. 22) as one of the stylized facts, "asset price bubbles tend to appear in periods of excitement about innovations".[2] Scheinkman (2014, p. 40) also points out "The increase in the price of assets during a bubble makes it easier to finance investments related to the new technologies". In fact, during the rapid increase in the stock prices of IT (information technology) related companies during the dot-com bubble era, there were numerous IPOs (see Scheinkman (2014, p. 18)), implying the numerous establishment of new companies. Unlike many other technologies, IT has a significant impact on production and innovation in many sectors, the characteristic that qualifies it as one of the "General Purpose Technologies (GPTs)" as defined by Bresnahan and Trajtenberg (1995).[3]

Based on these motivations, we construct a macro-finance model of intangible capital, with positive spillovers to the rest of the economy. We show that stochastic stock bubbles attached to intangible capital emerge in the process of spillover of technological innovation. More precisely, we adopt the innovation-driven growth model developed by Grossman and Helpman (1991a, Ch.3) to analyze the relationship between knowledge spillovers from knowledge-intensive sectors, such as IT, to other production factors (or sectors) and the emergence of stock bubbles. Our analysis reveals that as long as the state of UG persists, where the spillover effects of innovation are strong and unevenly spread across production factors, the economy temporarily deviates from the Balanced Growth Path (BGP) and exhibits

[2] Scheinkman (2014, p. 22) notes "The stock market bubble of the 1920s was driven primarily by the new technology stocks of the time, namely the automobile, aircraft, motion picture, and radio industries; the dotcom bubble has an obvious connection to Internet technology".

[3] Bresnahan and Trajtenberg (1995) characterizes GPTs as displaying three fundamental features: (i) pervasiveness (they spread to a wide range of sectors), (ii) improvement (they can continuously evolve), and (iii) innovation spanning (They enhance new secondary innovations). Another well-known example is the steam engine. As the construction of railways adopting steam engines advanced, a stock price bubble known as the "Railway Mania" emerged.

unbalanced growth dynamics. During this phase, the knowledge-intensive sector experiences the emergence of stock bubbles, where rising stock prices encourage the establishment of new firms and further innovations. These innovations, in turn, increase future wages, stock prices, and further innovations, creating a virtuous cycle. During this phase, the economy enjoys an era of high growth. However, once the state of BG arises, in which knowledge spillovers weaken and the effects of innovations become evenly spread across production factors, the economy returns to the BGP and the stock bubble bursts. While innovation slows down after the bubble bursts and the economy transitions into a low-growth era, the technologies developed during the bubble period persist, leading to a higher level of post-bubble GDP as the bubble period lasts longer. These results are consistent with the narrative"the relationship between bubbles and technological innovation suggests that some of these episodes may play a positive role in economic growth" highlighted by Scheinkman (2014, p. 40).

As the models in Sections 2 and 3 show, the dynamics with stochastic bubbles, which is characterized by unbalanced growth, can be seen as a temporary deviation from a balanced growth path in which asset prices equal the fundamentals, i.e., the expected present discounted value of future dividends.

Implication for macro-theory construction Our construction of a macro-finance model where unbalanced growth dynamics can temporarily occur provides a new perspective on the methodology of macro-theory construction because asset pricing implications change markedly. That is to say, as is well known as "Uzawa steady-state growth theorem", which is the heart of macro-theory construction with a balanced growth path (BGP), any growth model that produces a BGP is knife edge theory (Uzawa, 1961; Schlicht, 2006; Jones and Scrimgeour, 2008). Indeed, Grossman, Helpman, Oberfield, and Sampson (2017, p. 1306) clearly note "As with any model that generates balanced growth, knife-edge restrictions are required to maintain the balance". Under knife-edge conditions that generate balanced growth, in many cases, there is a single dynamic path that can be drawn with one stroke of the brush and along the dynamic path, the macroeconomy converges to a steady state characterized by balanced growth, in which asset prices and dividends grow at the same rate. As long as we construct a model in this way, it is assumed from the beginning of model construction that asset prices are equal to the fundamentals. What our paper shows is that even the slightest deviation from the knife-edge cases leads to markedly different implications for asset prices.[4] The macroeconomy temporarily takes a different dynamic path from the BGP, and in this transitional dynamics, stochastic bubbles emerge. Based on these insights, in Section 4.1, as the third main result, we will uncover the relationship between the Uzawa steady-state growth theorem and asset price bubbles.

2 Stochastic bubbles in toy model

To illustrate the essence of the emergence of stochastic bubbles, we present a toy model in endowment economies in which land price bubbles that are expected to collapse emerge as the unique equilibrium outcome. Then we identify the economic conditions for the emergence of stochastic bubbles.

2.1 The basic setup

We consider a standard two-period overlapping generations (OLG) model with two sectors, i.e., land and income-generating sectors, respectively. Time is indexed by $t = 0, 1, \dots$. In each period, young agents with a unit measure are born, and they live for two periods. Each young person is endowed with $e_t^{s_t}$ units of consumption goods (only) when young, where $s_t \in \{UG, BG\}$ denotes each state at date t. $e_t^{s_t}$ can be interpreted as the income the young receive by working in the income-generating sector. In the land sector, there is a fixed supply of land with X, and a unit of land produces $D_t^{s_t}$ units of consumption goods as dividends in each period. $e_t^{s_t}$ and $D_t^{s_t}$ capture productivities of the respective sectors. There are two states in the macroeconomy. One state of $s_t = UG$ is characterized by unbalanced growth, where the productivity growth rates of the two sectors are different, while the other state of $s_t = BG$ is characterized by balanced growth, where the productivity growth rates are equal.

Following Weil (1987), $s_t \in \{UG, BG\}$ follows the following Markov chain.

Assumption 1. *Letting s_t denote the state of the economy at time t, we have*

$$\Pr[s_{t+1} = UG \mid s_t = UG] = \pi \in (0, 1), \tag{2.1a}$$

$$\Pr[s_{t+1} = UG \mid s_t = BG] = 0. \tag{2.1b}$$

[4]We thank Joseph Stiglitz for this approach because one of the authors (Hirano) took a big hint through continuous discussions with him about the methodology in which take a standard model as it is and change only one part, which then leads to markedly different economic insights.

This assumption implies that the state of BG is an absorbing state.

The realizations of $D_t^{s_t} \in \{D_t^{UG}, D_t^{BG}\}$ and $e_t^{s_t} \in \{e_t^{UG}, e_t^{BG}\}$ are governed by the following assumptions.

Assumption 2.

$$D_{t+1}^{UG} = G_d' D_t^{UG}, \text{ and } D_{t+1}^{BG} = G_d D_t^{BG} \tag{2.2}$$

Assumption 3.

$$e_{t+1}^{UG} = G_a e_t^{UG}, \text{ and } e_{t+1}^{BG} = G_d e_t^{BG}. \tag{2.3}$$

Assumption 4.

$$G_a > 1, \text{ and } G_a > G_d' \geq G_d. \tag{2.4}$$

These assumptions imply that as long as the state of UG persists, the productivity growth rate of the income-generating sector is higher, and hence the incomes of the young grow faster than rents, exhibiting unbalanced growth. Once the probability $1 - \pi$ arises, the productivity growth rates are equal, exhibiting balanced growth, and therefore the income of the young generation and the land rents grow at the same rate. We will see these points again in (2.10).

In addition, we also consider the case where the initial date $t = 0$, $s_0 = UG$ and $e_0^{UG} > e_0^{BG}$ and $D_0^{UG} > D_0^{BG}$. In this case, when the state of $s_t = BG$ arises at date $t = T$, according to the assumptions 1, 2, 3, and 4, both D_T and e_T will decrease to levels of what they would have been if the macroeconomy had taken $s_t = BG$ all the way from the initial date 0.

Utility function The utility function of each young person is given by

$$u_t^{s_t} = \mathrm{E}_t[c_{t+1}^{s_{t+1}}], \tag{2.5}$$

where $u_t^{s_t}$ is expected utility at date t, and $\mathrm{E}_t[\cdot]$ is the expected value conditional on information available at date t, and $c_{t+1}^{s_{t+1}}$ is consumption when old.

The budget constraint The budget constraints of each agent in young and old periods are

$$P_t^{s_t} x_t^{s_t} = e_t^{s_t} \text{ and } c_{t+1}^{s_{t+1}} = (D_{t+1}^{s_{t+1}} + P_{t+1}^{s_{t+1}}) x_t^{s_t}, \tag{2.6}$$

where $x_t^{s_t}$ is the amount of land holdings at date t, and $P_t^{s_t}$ is the price of land at date t in each state.

From (2.5) and (2.6), we obtain indirect utility

$$u_t^{s_t} = R_t^{s_t} e_t^{s_t}, \tag{2.7}$$

where

$$R_t^{s_t} := \mathrm{E}_t \left[\frac{D_{t+1}^{s_{t+1}} + P_{t+1}^{s_{t+1}}}{P_t^{s_t}} \right]. \tag{2.8}$$

$R_t^{s_t}$ captures how society trades off resources between dates t and $t+1$. Because hypothetically if a social planner took resources from a young agent at date t, (s)he would require $R_t^{s_t}$ units of consumption goods to maintain the same expected utility.

2.2 Equilibrium

Each young person maximizes his expected utility (2.5) subject to (2.6) in each period.

The market for consumption goods clears at all dates.

$$C_t^{s_t} = e_t^{s_t} + D_t^{s_t} X, \tag{2.9}$$

$C_t^{s_t}$ is aggregate consumption at date t, and the land market clears, i.e., $x_t^{s_t} = X$.

By dividing both sides by $e_t^{s_t}$ in (2.9), we obtain

$$\frac{C_t^{s_t}}{e_t^{s_t}} = 1 + \frac{D_t^{s_t} X}{e_t^{s_t}}. \tag{2.10}$$

From assumptions 2 and 3, we learn that as long as the state of UG persists, endowments (incomes of the young) grow faster than land rents, and therefore the ratio of aggregate land rents to aggregate endowments decreases. Once the probability $1-\pi$ arises, endowments and rents grow at the same rate, and therefore the economy is back to balanced growth, in which the ratios of aggregate consumption and land rents to aggregate endowments are constant.

2.3 Dynamics

So long as the state of UG persists,

$$P_t^{UG} X = e_t^{UG}. \tag{2.11}$$

Hence, we obtain

$$\frac{P_{t+1}^{UG}}{P_t^{UG}} = \frac{e_{t+1}^{UG}}{e_t^{UG}} = G_a. \tag{2.12}$$

That is, land prices, pulled up by high endowment (income) growth, will rise.

On the other hand, conditional on $e_t^{s_t} = e_t^{BG}$,

$$P_t^{BG} = e_t^{BG}, \tag{2.13}$$

i.e., when the probability $1-\pi$ arises, the land price falls and afterward, it grows at the same rate of land rents (assumption 3).

Conditional on $e_t^{s_t} = e_t^{UG}$ at time t, (2.8) is now written as

$$R_t^{UG} = \frac{\pi D_{t+1}^{UG}}{P_t^{UG}} + \frac{(1-\pi)D_{t+1}^{BG}}{P_t^{UG}} + \frac{\pi P_{t+1}^{UG}}{P_t^{UG}} + \frac{(1-\pi)P_{t+1}^{BG}}{P_t^{UG}}. \tag{2.14}$$

Since D_t^{UG}, D_t^{BG}, P_t^{UG}, and P_t^{BG} follow according to (2.2), (2.11), and (2.13), each young person can compute the future values of each variable after date $t+1$ onward, as of time t. Therefore, they are known as of time t.

Similarly, conditional on $e_t^{s_t} = e_t^{BG}$ at time t, (2.8) is written as

$$R_t^{BG} = \frac{D_{t+1}^{BG}}{P_t^{BG}} + \frac{P_{t+1}^{BG}}{P_t^{BG}} = G_d\left(\frac{D_t^{BG}}{P_t^{BG}} + 1\right) > G_d. \tag{2.15}$$

Hence, land prices at time t when probability $1-\pi$ arises at time t is

$$P_t^{BG} = \frac{G_d D_t^{BG}}{R_t^{BG} - G_d}. \tag{2.16}$$

After time t onward, R_t^{BG} is constant because P_t^{BG} and D_t^{BG} grow at the same rate. Hence, once the state of the macroeconomy is back on the balanced growth path, P_t^{BG} equals the discounted present value of future rents, i.e., the fundamental value, and the price-rent ratio P_t^{BG}/D_t^{BG} is constant.

We are now ready to define stochastic bubbles that are expected to collapse. To examine them, we consider when the economy is in state UG at time t.

Conditional on $e_t^{s_t} = e_t^{UG}$ at time t, the fundamental value of land prices at time t is defined as

$$V_t^{UG} := \sum_{n=1}^{\infty} \frac{\pi^{n-1}\left[\pi D_{t+n}^{UG} + (1-\pi)D_{t+n}^{BG} + (1-\pi)P_{t+n}^{BG}\right]}{\prod_{j=0}^{n-1} R_{t+j}^{UG}}, \tag{2.17}$$

i.e., (2.17) equals the expected discounted present value of future dividends, as of time t. Recall that P_t^{BG} is equal to the discounted value of future rents.

Solving (2.14) for P_t^{UG} and iterating forward yields

$$P_t^{UG} = \sum_{n=1}^{N} \frac{\pi^{n-1}\left[\pi D_{t+n}^{UG} + (1-\pi)D_{t+n}^{BG} + (1-\pi)P_{t+n}^{BG}\right]}{\prod_{j=0}^{n-1} R_{t+j}^{UG}} + \frac{\pi^N P_{t+N}^{UG}}{\prod_{j=0}^{N-1} R_{t+j}^{UG}}. \tag{2.18}$$

As $N \to \infty$ in (2.18), we obtain $P_t^{UG} = V_t^{UG} + B_t$, where we define land price bubbles as

$$B_t := \lim_{N\to\infty} \frac{\pi^N P_{t+N}^{UG}}{\prod_{j=0}^{N-1} R_{t+j}^{UG}} \geq 0. \tag{2.19}$$

That is, a land price bubble is equal to the difference between the market price of land and its fundamental value. By definition, there is no bubble at time t if and only if the no-bubble condition holds. That is,

$$\lim_{N\to\infty} \frac{\pi^N P_{t+N}^{UG}}{\prod_{j=0}^{N-1} R_{t+j}^{UG}} = 0. \tag{2.20}$$

The economic meaning of the bubble component B_t in (2.19) is that it captures a speculative aspect, that is, agents buy land now for the purpose of resale in the future, rather than for the purpose of receiving dividends. When the no-bubble condition (2.20) holds, the aspect of speculation becomes negligible and land prices are determined only by factors that are backed in equilibrium, namely the expected future dividends. On the other hand, if $B_t > 0$, equilibrium land prices contain a speculative aspect. They are priced above the expected discounted present value of the rents received.

Note that the deterministic case without aggregate uncertainty can be described by setting $\pi = 1$. Hence, the state of the macroeconomy after the event of $1-\pi$ corresponds to the case where $\pi = 1$ and UG is replaced by BG in (2.19) and (2.20). The no-bubble condition (2.20) is satisfied, i.e., there is no bubble after the probability $1-\pi$ arises.

2.4 Emergence of stochastic bubbles

In this section, we will show that stochastic land bubbles that are expected to collapse emerge as the unique equilibrium outcome.

Considering $P_{t+n}^{BG} = (G_d)^n e_t^{BG}$, $D_{t+n}^{UG} = (G_d')^n D_t^{UG}$, and $D_{t+n}^{BG} = (G_d)^n D_t^{BG}$, (2.17) becomes

$$\frac{V_t^{UG}}{D_t^{UG}} = \sum_{n=1}^{\infty} \frac{\left[\pi + (1-\pi)(G_d/G'_d)^{t+n}(D_0^{BG}/D_0^{UG} + e_0^{BG}/D_0^{UG})\right](G'_d)^n}{\prod_{j=0}^{n-1} R_{t+j}^{UG}}. \tag{2.21}$$

When $G'_d > G_d$, by taking the limit, we obtain

$$\lim_{t\to\infty} \frac{V_t^{UG}}{D_t^{UG}} = \frac{\pi G'_d}{R^{UG} - \pi G'_d} < \infty, \tag{2.22}$$

because $R^{UG} := \lim_{t\to\infty} R_t^{UG} = \frac{\pi P_{t+1}^{UG}}{P_t^{UG}} = \pi G_a > \pi G'_d$. When $G'_d = G_d$, the numerator in (2.22) is replaced by $\pi + (1-\pi)(D_0^{BG}/D_0^{UG} + e_0^{BG}/D_0^{UG})$. In either case, the fundamental value-rent ratio will be finite as long as the state of UG persists.

On the other hand, rearranging (2.14) yields

$$\frac{P_t^{UG}}{D_t^{UG}} = \frac{G'_d}{R_t^{UG} - \frac{\pi P_{t+1}^{UG}}{P_t^{UG}} - \frac{(1-\pi)D_{t+1}^{BG}}{P_t^{UG}} - \frac{(1-\pi)P_{t+1}^{BG}}{P_t^{UG}}}. \tag{2.23}$$

Then, we obtain

$$\lim_{t\to\infty} \frac{P_t^{UG}}{D_t^{UG}} = \frac{G'_d}{R_t^{UG} - \frac{\pi P_{t+1}^{UG}}{P_t^{UG}} - \frac{(1-\pi)D_{t+1}^{BG}}{P_t^{UG}} - \frac{(1-\pi)P_{t+1}^{BG}}{P_t^{UG}}} = \infty, \tag{2.24}$$

where $\lim_{t\to\infty} R_t^{UG} = \frac{\pi P_{t+1}^{UG}}{P_t^{UG}}$, $\lim_{t\to\infty} \frac{D_{t+1}^{BG}}{P_t^{UG}} = \lim_{t\to\infty} \frac{G_d D_t^{BG}}{P_t^{UG}} = 0$, and $\lim_{t\to\infty} \frac{P_{t+1}^{BG}}{P_t^{UG}} = \lim_{t\to\infty} \frac{G_d e_t^{BG}}{P_t^{UG}} = 0$ (P_t^{UG} grows at a faster rate than D_t^{BG} and e_t^{BG} because $G_a > G_d$). From (2.24), we learn that the price-rent ratio will increase without bound so long as the state of UG persists.

The comparison between (2.22) and (2.24) shows that the equilibrium price of land will eventually exceed its fundamental value and contain a bubble. Moreover, once land prices are expected to contain bubbles in the future, land bubbles will be included even in the current period. In other words, by the backward induction argument, $P_t^{UG} > V_t^{UG}$ at all dates.

For instance, consider a special case in which $G_d = G'_d = 1$. In this case, D^{UG} and D^{BG} are constant, and both e^{BG} and the land price at the time of bubble burst P^{BG} are also constant. On the other hand, P_t^{UG} will increase at the rate of G_a as long as the probability π persists. This special case clearly shows that when the probability $1-\pi$ arises, the land bubble not only collapses, but also the price will fall sharply as the bubble period lasts longer.

This toy model tells us that so long as the state of UG persists and endowments (incomes of the young) grow at a faster rate than rents, the macroeconomy exhibits unbalanced growth, along which land prices will rise, including bubbles, and the price-rent ratio will increase divergently. Once the probability $1-\pi$ arises, then the land bubbles collapse and the macroeconomy returns to balanced growth, where the price-rent ratio is stable.

2.5 Three economic implications to be drawn

From the above toy model, we can draw three economic implications.

1. First, the toy model we have presented implies that the dynamics of land price bubbles with a divergent increase in the price-rent ratio can be seen as a temporary deviation from the balanced growth path where land prices equal the discounted present value of future dividends, i.e., the fundamentals.

2. Second, since land prices are uniquely determined, land price bubbles that are expected to collapse emerge as the unique equilibrium outcome.

3. Third, we can derive the economic conditions under which stochastic bubbles emerge as the equilibrium outcome. That is to say,

   (a) P_t^{UG} increases faster rate than land rents, D_t^{UG}, i.e., unbalanced growth occurs, and

   (b) P_t^{UG} increases at a faster than D_t^{BG}, which implies that the growth rate of income in the state of unbalanced growth is higher than that in the state of balanced growth, and

   (c) P_t^{UG} grows faster than P_t^{BG}, which implies that the longer the state of UG persists, the sharper the fall in land prices.

   If and only if conditions (a), (b), and (c) are simultaneously satisfied, we will obtain $\pi \frac{P_{t+1}^{UG}}{P_t^{UG}} \sim R_t^{UG}$ in (2.14) asymptotically, in which case the no-bubble condition (2.20) will be violated.

   Two points need to be stated. First, unlike the case of deterministic bubbles in which condition (a) alone is sufficient to generate bubbles, conditions (b) and (c) are also needed to generate stochastic bubbles. Second, a regime-switching model with π and $1-\pi$ does not mean that stochastic bubbles will always emerge because conditions (a), (b), and (c) are not always satisfied. A natural next question would be if there exists a plausible economic model with investment and production in which all three conditions are simultaneously satisfied. In Section 3, we will present such a model.

2.6 Multiple savings vehicles

So far, land is the only saving vehicle. A natural question would be what happens with multiple savings vehicles. In this section, we will show that our argument in the previous section will hold even when there are multiple means of saving.

To address this issue, we introduce another saving vehicle. We consider another dividend-paying asset, e.g., a stock market index with outstanding shares normalized to S. Stock market index per share produces dividends $r_t^{s_t}$ in every period depending on each state, and its price at date t is $Q_t^{s_t}$.

Considering the no-arbitrage equation between the two assets, R_t^{UG} is now written as

$$\begin{aligned} R_t^{UG} &= \frac{\pi D_{t+1}^{UG}}{P_t^{UG}} + \frac{(1-\pi)D_{t+1}^{BG}}{P_t^{UG}} + \frac{\pi P_{t+1}^{UG}}{P_t^{UG}} + \frac{(1-\pi)P_{t+1}^{BG}}{P_t^{UG}} \\ &= \frac{\pi r_{t+1}^{UG}}{Q_t^{UG}} + \frac{(1-\pi)r_{t+1}^{BG}}{Q_t^{UG}} + \frac{\pi Q_{t+1}^{UG}}{Q_t^{UG}} + \frac{(1-\pi)Q_{t+1}^{BG}}{Q_t^{UG}}. \end{aligned} \tag{2.25}$$

Rearranging (2.25) yields

$$\begin{aligned} P_t^{UG} &= \frac{\pi D_{t+1}^{UG}}{R_t^{UG}} + \frac{(1-\pi)D_{t+1}^{BG}}{R_t^{UG}} + \frac{\pi P_{t+1}^{UG}}{R_t^{UG}} + \frac{(1-\pi)P_{t+1}^{BG}}{R_t^{UG}} \\ \Longleftrightarrow P_t^{UG}X &= \frac{\pi D_{t+1}^{UG}X}{R_t^{UG}} + \frac{(1-\pi)D_{t+1}^{BG}X}{R_t^{UG}} + \frac{\pi P_{t+1}^{UG}X}{R_t^{UG}} + \frac{(1-\pi)P_{t+1}^{BG}X}{R_t^{UG}} \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} Q_t^{UG} &= \frac{\pi r_{t+1}^{UG}}{R_t^{UG}} + \frac{(1-\pi)r_{t+1}^{BG}}{R_t^{UG}} + \frac{\pi Q_{t+1}^{UG}}{R_t^{UG}} + \frac{(1-\pi)Q_{t+1}^{BG}}{R_t^{UG}} \\ \Longleftrightarrow Q_t^{UG}S &= \frac{\pi r_{t+1}^{UG}S}{R_t^{UG}} + \frac{(1-\pi)r_{t+1}^{BG}S}{R_t^{UG}} + \frac{\pi Q_{t+1}^{UG}S}{R_t^{UG}} + \frac{(1-\pi)Q_{t+1}^{BG}S}{R_t^{UG}}. \end{aligned} \tag{2.27}$$

Adding (2.26) and (2.27) together yields

$$\begin{aligned} P_t^{UG}X + Q_t^{UG}S = &\frac{\pi\left(D_{t+1}^{UG}X + r_{t+1}^{UG}S\right)}{R_t^{UG}} + \frac{(1-\pi)\left(D_{t+1}^{BG}X + r_{t+1}^{BG}S\right)}{R_t^{UG}} \\ &+ \frac{\pi\left(P_{t+1}^{UG}X + Q_{t+1}^{UG}S\right)}{R_t^{UG}} + \frac{(1-\pi)\left(P_{t+1}^{BG}X + Q_{t+1}^{BG}S\right)}{R_t^{UG}}. \end{aligned} \tag{2.28}$$

We learn that if we compare (2.14) with (2.28), they are parallel. That is, aggregate assets of land and shares can be considered as if they were a single asset. Then, the argument in the previous section i.e., conditions (a), (b), and (c), can be applied directly. That is, stochastic bubbles in the aggregate value of land and stocks emerge as the unique equilibrium outcome if (i) $P_t^{UG}X + Q_t^{UG}S$ grows faster than $D_t^{UG}X + r_t^{UG}S$, and (ii) $P_t^{UG}X + Q_t^{UG}S$ grows faster than $D_t^{BG}X + r_t^{BG}S$, and (iii) $P_t^{UG}X + Q_t^{UG}S$ grows faster than $P_{t+1}^{BG}X + Q_{t+1}^{BG}S$. For example, if the growth rates of r_t^{UG} and r_t^{BG} are equal to G'_d and G_d, respectively, in which case the economy returns to balanced growth after probability $1 - \pi$ arises, or are lower than G_a, these three conditions are simultaneously satisfied and, therefore, stochastic bubbles in land and stock prices simultaneously emerge.[5]

As we can see from this analysis, without loss of generality, the same argument can be applied when extended to the N-asset model. From the no-arbitrage condition across N assets, by bundling N assets together as a single asset, macro bubbles can be described with a single asset model. Hence, in the next section, to avoid complexity and illustrate the point, we focus on a model with a single asset.

3 Innovation and stochastic stock bubbles

So far, we have presented an example that shows the existence of stochastic bubbles in endowment economies. In this section, we construct a full-fledged macroeconomic model with intangible capital and production, and show that stochastic stock price bubbles attached to intangible capital emerge as the equilibrium outcome. It will also be shown that the dynamics with stock price bubbles associated with innovation is a temporary deviation from balanced growth in which stock prices reflect the fundamentals. The model we present is a growth model with innovation.

3.1 The basic setup

The essential structure of the model is similar to the variety expansion model of Grossman and Helpman (1991a, Ch.3). To illustrate the key mechanism of how stochastic stock bubbles emerge, we reformulate their model into a two-period overlapping generations (OLG) framework.

[5]Note that even though the bubble sizes on individual assets are indeterminate, the total size of the bubble is determinate, and hence the consumption allocation is identical regardless of the size of the bubble attached to each asset. This argument is the same as the "bubble substitution" argument in Tirole (1985, §5).

Time is indexed by $t = 0, 1, \ldots$. There are two types of individuals. In each period, a continuum measure H of skilled labor and a continuum measure L of unskilled labor are born, who live for two periods. Each individual has one unit of time only when young. Each skilled labor optimally allocates one unit of time to each of the R&D activities and labor for the production of the knowledge-intensive intermediate goods. On the other hand, each unskilled labor inelastically supplies one unit of time to labor in the consumer goods sector.

3.2 Consumption goods sector

There is a representative competitive firm that produces the consumption goods. The aggregate production function is given by

$$Y_t^{s_t} = \left[\alpha(A_t^{s_t}L)^{1-\rho} + (1-\alpha)(Z_t^{s_t}X_t^{s_t})^{1-\rho}\right]^{\frac{1}{1-\rho}}, \tag{3.1}$$

where $Y_t^{s_t}$ is output of consumption goods when the state of the economy is $s_t \in \{UG, BG\}$ at date t, $A_t^{s_t}$ is the productivity level of unskilled labor L in the state s_t at date t, $X_t^{s_t}$ is input of knowledge-intensive goods when the state of economy is s_t at date t, $Z_t^{s_t}$ is the productivity level of $X_t^{s_t}$ in the state s_t at date t, and $\rho > 0$ and $\alpha \in (0, 1)$ are parameters. $1/\rho$ is elasticity of substitution between $A_t^{s_t}L$ and $Z_t^{s_t}X_t^{s_t}$. If we set $\alpha = 0$, the production function (3.1) simplifies to the one presented in Grossman and Helpman (1991a, Ch.3).[6]

As in Section 2, we introduce aggregate uncertainty to describe stochastic bubbles. Let $s_t \in \{UG, BG\}$ represents the state of the economy at date t. When the economy is in state $s_t = UG$, the productivity growth rates of the production factors $A_t^{s_t}L$ and $Z_t^{s_t}X_t^{s_t}$ differ , i.e., unbalanced growth occurs. On the other hand, when $s_t = BG$, the productivity growth is equal across production factors so that balanced growth is achieved. The conditions of the specific parameters for each state are explained in detail in Section 3.8. Suppose that the initial state is $s_0 = UG$. The evolution of state follows the assumption 1.

We choose the consumption goods as numeraire. From the first order condi-

[6]Strictly following the setting of Grossman and Helpman (1991a, Ch.3), (3.1) would be defined as a utility function rather than a production function. When interpreted as a utility function, households derive utility from two types of goods, $A_t^{s_t}L$ and $Z_t^{s_t}X_t^{s_t}$. In the original Grossman and Helpman (1991a, Ch.3) framework, this corresponds to $\alpha = 0$, meaning that utility is derived from a single composite good $Z_t^{s_t}X_t^{s_t}$.

tions for the profit maximization problem, we obtain

$$w_{L,t}^{s_t} = \alpha A_t^{s_t} \left(\frac{Y_t^{s_t}}{A_t^{s_t} L} \right)^{\rho} \tag{3.2}$$

and

$$Q_t^{s_t} = (1-\alpha) Z_t^{s_t} \left(\frac{Y_t^{s_t}}{Z_t^{s_t} X_t^{s_t}} \right)^{\rho}, \tag{3.3}$$

where $w_{L,t}^{s_t}$ is the wage rate of labor L in the state s_t at date t and $Q_t^{s_t}$ is the price of the good $X_t^{s_t}$ in the state s_t at date t. Note that $Q_t^{s_t}$ here is different from that in Section 2.6.

3.3 Knowledge-intensive goods sector

There is a representative competitive firm that produces the knowledge-intensive goods. The aggregate production function is given by

$$X_t^{s_t} = \left(\int_0^{n_t^{s_t}} [x_t^{s_t}(i)]^{\gamma} di \right)^{\frac{1}{\gamma}}, \tag{3.4}$$

where $x_t^{s_t}(i)$ is input of differentiated intermediate goods i in the state s_t at date t, $n_t^{s_t}$ is number of varieties of intermediate goods in the state s_t at date t, and $\gamma \in (0,1)$ is a parameter. As will be explained in Section 3.5, $n_t^{s_t}$ is endogenously determined and grows as a result of innovation. In other words, new goods are developed through innovation. In this context, $n_t^{s_t}$ represents the total variety of goods created by past innovations and can be viewed as the stock of ideas (knowledge) in the economy.

From the first order condition for the profit maximization problem, we obtain a factor demand function:

$$x_t^{s_t}(i) = \left(\frac{q_t^{s_t}(i)}{Q_t^{s_t}} \right)^{\frac{-1}{1-\gamma}} X_t^{s_t}, \tag{3.5}$$

where $q_t^{s_t}(i)$ is the price of intermediate good i in the state s_t at date t. Substituting (3.5) into (3.4) we obtain the price of $X_t^{s_t}$:

$$Q_t^{s_t} = \left(\int_0^{n_t^{s_t}} [q_t^{s_t}(i)]^{\frac{-\gamma}{1-\gamma}} di \right)^{-\frac{1-\gamma}{\gamma}}. \tag{3.6}$$

3.4 Differentiated intermediate goods sector

The differentiated intermediate good $x_t^{s_t}(i)$ is produced under monopolistic competition. One unit of the intermediate good is produced by the input of one unit of time by skilled labor. Facing the demand function (3.5), the unique producer of variety i maximizes profits

$$D_t^{s_t}(i) = q_t^{s_t}(i)x_t^{s_t}(i) - w_{H,t}^{s_t}x_t^{s_t}(i), \tag{3.7}$$

where $w_{H,t}^{s_t}$ is wage rate of skilled labor in the state s_t, by charging a price $q_t^{s_t}(i) = w_{H,t}^{s_t}/\gamma$. Then, all varieties are priced equally at $q_t^{s_t}$ where

$$q_t^{s_t} = \frac{w_{H,t}^{s_t}}{\gamma}. \tag{3.8}$$

Since firms price equally, the output of goods is also equal at $x_t^{s_t}$ across varieties where

$$x_t^{s_t} = (n_t^{s_t})^{\frac{-1}{\gamma}} X_t^{s_t} \tag{3.9}$$

and the profit is also equal at $D_t^{s_t}$ across varieties where

$$D_t^{s_t} = \left(\frac{1-\gamma}{\gamma}\right) w_{H,t}^{s_t}x_t^{s_t}. \tag{3.10}$$

3.5 R&D sector

An skilled labor who devotes $\tau_t^{s_t} \in [0,1]$ units of time to R&D activities in the state s_t at date t creates $an_t^{s_t}\tau_t^{s_t}$ units of new differentiated intermediate products, where $a > 0$ is a R&D productivity parameter. The newly invented varieties are fully protected by patents for an infinite period. Therefore, as analyzed in section 3.4, the sole producer of each variety holds monopoly power and earns the monopoly profit $D_t^{s_t}$ indefinitely. The stock price of a firm, $P_t^{s_t}$, can also be interpreted as the price of an idea or patent, i.e., intangible capital.[7]

The term $n_t^{s_t}$ in R&D production function $an_t^{s_t}\tau_t^{s_t}$ captures knowledge spillover from the past innovations to the current innovations, and, as is well known in the literature, this linearity assumption of knowledge spillover is necessary for producing endogenous growth.

[7] It is possible to consider not patent protection for ideas regarding the production methods of goods, but instead to keep the ideas as trade secrets. If it is assumed that the trade secret can fully protect the idea, then the analysis of the model would not differ. Therefore, in general, $P_t^{s_t}$ can be considered as the price of the idea regarding the production method of the goods.

3.6 Household

Utility function of skilled labors and unskilled labors is given by

$$u_{j,t}^{s_t} = \mathrm{E}_t[c_{j,t+1}^{s_t}], \tag{3.11}$$

where $u_{j,t}^{s_t}$ is utility level in the state s_t at date t, $\mathrm{E}_t[\cdot]$ is the expected value conditional on information available at date t, and $c_{j,t+1}^{s_t}$ is consumption when old. Here, $j \in \{H, L\}$, where H refers to skilled labors and L refers to unskilled labors.

The budget constraints in young and old periods are

$$P_t^{s_t} n_{t+1}^{s_{t+1}} m_{j,t}^{s_t} = W_{j,t}^{s_t} \text{ and } c_{j,t+1}^{s_{t+1}} = (D_{t+1}^{s_{t+1}} + P_{t+1}^{s_{t+1}}) n_{t+1}^{s_{t+1}} m_{j,t}^{s_t}, \tag{3.12}$$

where $P_t^{s_t}$ and $D_t^{s_t}$ are stock price of the firms and dividend per share in the state s_t at date t. We assume the law of one price, as in ordinary macroeconomic models, i.e., if the dividend and fundamental value of stocks are the same, the price will be the same. Unlike Section 2, $D_t^{s_t}$ is now endogenously determined and state dependent. $n_{t+1}^{s_{t+1}}$ is a continuum measure of firms established before date $t+1$, including the new firms established at date t, $m_{j,t}^{s_t} \in [0, 1]$ is asset holding rate of the individual $j \in \{H, L\}$, $W_{j,t}^{s_t}$ is income level of individual $j \in \{H, L\}$ in the state s_t at date t.

From (3.11) and (3.12), we obtain indirect utility

$$u_{j,t}^{s_t} = R_t^{s_t} W_{j,t}^{s_t}, \tag{3.13}$$

where $R_t^{s_t}$ is given by (2.8).

Skilled labor optimally chooses $\tau_t^{s_t}$ to maximize the expected utility. The income level of skilled labor is given by $W_{H,t}^{s_t} = a n_t^{s_t} \tau_t^{s_t} P_t^{s_t} + (1 - \tau_t^{s_t}) w_{H,t}^{s_t}$. Then, the interior solution for the utility maximization problem gives

$$a n_t^{s_t} P_t^{s_t} = w_{H,t}^{s_t}. \tag{3.14}$$

The choice of $\tau_t^{s_t}$ is indifferent under (3.14). Suppose each skilled labor chooses the same R&D activity level $\tau_t^{s_t}$, then the evolution of the number of firms is

$$n_{t+1}^{s_{t+1}} - n_t^{s_t} = a n_t^{s_t} \tau_t^{s_t} H. \tag{3.15}$$

We can see that the number of varieties $n_t^{s_t}$ increases over time if and only if $\tau_t^{s_t} > 0$.

Since skilled labors work $1 - \tau_t^{s_t}$ units of time, the following labor market clearing condition holds:

$$n_t^{s_t} x_t^{s_t} = (1 - \tau_t^{s_t})H. \tag{3.16}$$

3.7 Asset market

Aggregating the first equation in (3.12) over all skilled and unskilled labors, and imposing the equilibrium condition $m_{H,t}^{s_t} H + m_{L,t}^{s_t} L = 1$, we obtain

$$[a n_t^{s_t} \tau_t^{s_t} P_t^{s_t} + (1 - \tau_t^{s_t}) w_{H,t}^{s_t}]H + w_{L,t}^{s_t} L = P_t^{s_t} n_{t+1}^{s_{t+1}}. \tag{3.17}$$

The first and second terms on the left-hand side represent the aggregate savings of skilled labors and unskilled labors, respectively, and the right-hand side represents the aggregate supply of firms' stock.

3.8 Productivity growth of production factors

From (3.9) and (3.16), the aggregate output of knowledge-intensive goods is

$$X_t^{s_t} = (n_t^{s_t})^{\frac{1-\gamma}{\gamma}} (1 - \tau_t^{s_t})H. \tag{3.18}$$

Following (Benassy, 1996, 1998), suppose

$$Z_t^{s_t} = (n_t^{s_t})^{\phi^{s_t}} (n_t^{s_t})^{-\frac{1-\gamma}{\gamma}}, \tag{3.19}$$

where $\phi^{s_t} > 0$ is a parameter in the state s_t, removing the role of the love-of-variety elasticity from the parameter γ. Since $Z_t^{s_t} X_t^{s_t} = (n_t^{s_t})^{\phi^{s_t}} (1 - \tau_t^{s_t})H$ holds, the parameter ϕ^{s_t} purely measures the extent to which additional variety from innovation increases the productivity of knowledge-intensive goods, that is, it measures the degree of love-of-variety.[8]

Suppose that

$$A_t^{s_t} = (n_t^{s_t})^{\psi^{s_t}}, \tag{3.20}$$

where $\psi^{s_t} > 0$, implying that there are knowledge spillovers from past innovations to the productivity of unskilled labors L.

[8] By redefining $\phi^{s_t} := (1 - \gamma^{s_t})/\gamma^{s_t}$ and setting $Z_t^{s_t} = 1$, instead of adopting the assumption (3.19) made by (Benassy, 1996, 1998), where γ^{s_t} is a parameter value of γ in the state s_t, our results will not change. However, if the assumption (3.19) are not adopted, any change in ϕ^{s_t} would simultaneously result in a change in γ^{s_t}, thereby altering price (3.8) and dividends (3.10) as well, which complicates the interpretation of the model further.

Then, from (3.1), (3.18), (3.19), and (3.20), aggregate output of consumption goods is

$$Y_t^{s_t} = \left[\alpha[(n_t^{s_t})^{\psi^{s_t}} L]^{1-\rho} + (1-\alpha)[(n_t^{s_t})^{\phi^{s_t}}(1-\tau_t^{s_t})H]^{1-\rho}\right]^{\frac{1}{1-\rho}}. \tag{3.21}$$

Therefore, the parameters ϕ^{s_t} and ψ^{s_t} determine the productivity growth rates of the two production factors $A_t^{s_t} L$ and $Z_t^{s_t} X_t^{s_t}$.

In the state $s_t = UG$, we assume

$$\phi^{UG} \neq \psi^{UG}, \tag{3.22}$$

where ϕ^{UG} and ψ^{UG} are the values of ϕ^{s_t} and ψ^{s_t} when $s_t = UG$, respectively. Under (3.22), when innovation generates new technologies, its spillover effects are unevenly spread across the two production factors, and hence their productivity growth rates differ as long as the state of UG persists. We refer to this situation as *unbalanced growth*.

Conversely, in the state $s_t = BG$, we suppose

$$\phi^{BG} = \psi^{BG} \tag{3.23}$$

holds, where ϕ^{BG} and ψ^{BG} are the new parameter values of ϕ^{s_t} and ψ^{s_t} after drawing the probability $1-\pi$. The condition (3.23) ensures that the productivity growth rates of the two production factors are equal, allowing the existence of the balanced growth path (BGP), along which the ratios of aggregate consumption and aggregate investment in the R&D activity to GDP, and the ratio of aggregate dividends from stocks to GDP are constant.

We impose the following condition on the parameter values.

Assumption 5.

$$\psi^{UG} > \phi^{BG} = \psi^{BG}. \tag{3.24}$$

(3.22) and assumption 5 imply that as long as the state of UG persists, the spillover effects of innovation are high and spread unevenly across the two production factors. However, once the state of UG ends, the spillover effects of innovation decrease and spread evenly throughout the economy.

Note that (3.23) is a knife edge condition, but as noted in the introduction, any growth model with BGP is knife edge theory.

3.9 Equilibrium dynamics of knowledge $n_t^{s_t}$

From (3.15), the growth rate of the number of varieties $n_t^{s_t}$ is written as

$$G_{n,t}^{s_t} := \frac{n_{t+1}^{s_{t+1}}}{n_t^{s_t}} = 1 + a\tau_t^{s_t} H. \tag{3.25}$$

The growth rate increases as $\tau_t^{s_t}$ increases. In the following, we will explain how $\tau_t^{s_t}$ evolves.

The equilibrium condition of the asset market determines the interior level of $\tau_t^{s_t}$ for a given $n_t^{s_t}$ (see Appendix A.1 for the derivation):

$$f(\tau_t^{s_t}, n_t^{s_t}) = \underbrace{\left(\frac{1}{aH}\right)}_{\frac{n_t^{s_t} P_t^{s_t}}{w_{H,t}^{s_t} H}}, \tag{3.26}$$

where

$$f(\tau_t^{s_t}, n_t^{s_t}) := \underbrace{1 - \tau_t^{s_t}}_{\frac{w_{H,t}^{s_t}(1-\tau_t^{s_t})H}{w_{H,t}^{s_t} H}} + \underbrace{\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\gamma}\right)\left(\frac{(n_t^{s_t})^{\phi^{s_t}} H}{(n_t^{s_t})^{\psi^{s_t}} L}\right)^{\rho-1} (1 - \tau_t^{s_t})^{\rho}}_{\frac{w_{L,t}^{s_t} L}{w_{H,t}^{s_t} H}}. \tag{3.27}$$

The first term of (3.27) comes from the aggregate wage income of skilled labor, while the second term comes from the aggregate wage income of unskilled labor. These incomes flow into the stock market to purchase existing shares, as represented by the right hand side of (3.26).

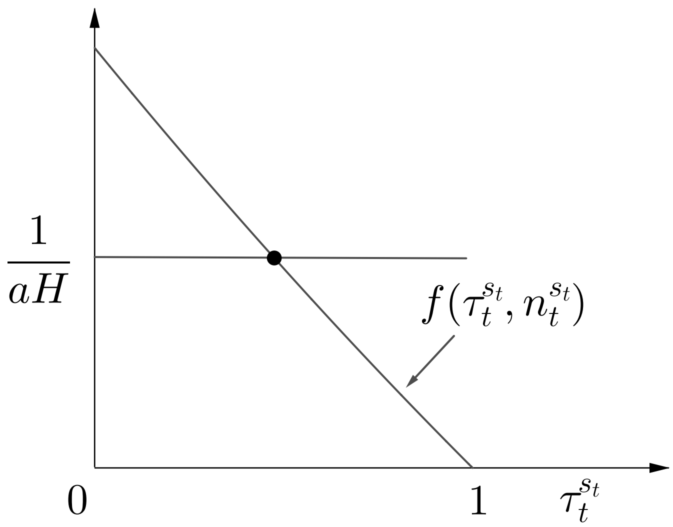

Figure 1: Solution to equation (3.26) when $f(0, n_t^{s_t}) > \frac{1}{aH}$ holds.

Figure 1 shows an interior solution of $\tau_t^{s_t}$. In the state of UG, if $(\phi^{UG}-\psi^{UG})(\rho-1)>0$, the second term on LHS of (3.27) increases as n_t^{UG} increases for a given τ_t^{UG}. Therefore, τ_t^{UG} increases over time and converges to 1.

To explain intuition, assume $\phi^{UG}>\psi^{UG}$ and $\rho>1$. This implies that the growth in n_t^{UG} due to innovation accelerates the growth of knowledge-intensive goods $Z_t^{UG}X_t^{UG}$ faster than effective labor $A_t^{UG}L$. Since $\rho>1$, the elasticity of substitution between factors $1/\rho$ is less than 1, indicating a complementary relationship. As a result, the faster growth of $Z_t^{UG}X_t^{UG}$ increases demand for its complement, labor L, raising the wages of unskilled labor relative to those of skilled labor. This increased wage flows into the stock market, driving up stock prices, which in turn stimulates R&D activities and raises τ_t^{UG}.

Next, assume $\phi^{UG}<\psi^{UG}$ and $\rho<1$. Here, the growth in n_t^{UG} causes labor $A_t^{UG}L$ to grow faster than knowledge-intensive goods $Z_t^{UG}X_t^{UG}$. With $\rho<1$, the elasticity of substitution exceeds 1, indicating a substitutional relationship. In this case, faster growth of productivity of labor boosts its labor demand, increasing unskilled labor's wages relative to those of skilled labor. Then, this increased wage flows into the stock market, driving up stock prices, which, in turn, raises τ_t^{UG}.[9]

Finally, consider the state BG where $(\phi^{BG}-\psi^{BG})(\rho-1)=0$ holds. The economy transitions to the BGP immediately without transitional dynamics. On the BGP, according to (3.26) and the condition (3.23), the interior equilibrium value of τ_t^{BG} is determined by

$$1-\tau_t^{BG}+\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\gamma}\right)\left(\frac{H}{L}\right)^{\rho-1}(1-\tau_t^{BG})^{\rho}=\frac{1}{aH}, \tag{3.28}$$

i.e., $\tau_t^{BG}=\tau^{BG}\in(0,1)$ is positive constant on the BGP.[10] The rate of productivity growth of the two factors of production is the same.

3.10 Emergence of stochastic stock bubbles

We are now ready to show the emergence of stochastic bubbles. For this purpose, we derive the normalized stock price and dividend, which allows us to show the emergence of stochastic stock bubbles intuitively. We consider a situation in which

[9]For the case where $(\phi^{UG}-\psi^{UG})(\rho-1)<0$, we can similarly derive the intuition.

[10]Note that the interior $\tau^{BG}\in(0,1)$ is ensured by the condition

$$1+\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\gamma}\right)\left(\frac{H}{L}\right)^{\rho-1}>\frac{1}{aH}.$$

If this condition is not satisfied, $\tau^{BG}=0$ holds, i.e., there are no R&D activities.

until date $t \in [0, T)$, $s_t = UG$ and the macroeconomy exhibits unbalanced growth, and after date $t \geq T$, $s_t = BG$ and the macroeconomy is back to balanced growth.

According to (3.9), (3.10), and (3.16), we obtain the normalized dividend

$$d_t^{s_t} := \frac{n_t^{s_t} D_t^{s_t}}{w_{H,t}^{s_t}} = \left(\frac{1-\gamma}{\gamma}\right)(1-\tau_t^{s_t})H. \tag{3.29}$$

Rewriting (3.14), we obtain the normalized stock price

$$p_t^{s_t} := \frac{n_t^{s_t} P_t^{s_t}}{w_{H,t}^{s_t}} = \frac{1}{a}. \tag{3.30}$$

Suppose that the current state is $s_t = UG$. Then using (3.29) and (3.30), the no-arbitrage condition (2.8) can be written as

$$R_t^{UG} p_t^{UG} = \pi(d_{t+1}^{UG} + p_{t+1}^{UG})\left(\frac{n_t^{UG}}{n_{t+1}^{UG}}\frac{w_{H,t+1}^{UG}}{w_{H,t}^{UG}}\right) + (1-\pi)(d_{t+1}^{BG} + p_{t+1}^{BG})\left(\frac{n_t^{UG}}{n_{t+1}^{BG}}\frac{w_{H,t+1}^{BG}}{w_{H,t}^{UG}}\right). \tag{3.31}$$

By solving (3.31) for p_t^{UG} and iterating it forward, we obtain equations corresponding to (2.17), (2.18), and (2.19), respectively, in terms of the normalized price and fundamental value. Note that D_t in (2.17) and (2.18) is now determined endogenously and state dependent.[11]

To prove the emergence of stochastic bubbles, we will take two steps. First, we will show that after $t \geq T$, $P_t^{BG} = V_t^{BG}$. Then, given this result, we will consider the economy when $s_t = UG$ at time $t < T$ and show $P_t^{UG} > V_t^{UG}$.

When the economy draws the probability $1-\pi$ in period T, the economy transitions to the BGP. On the BGP, the no-arbitrage condition in terms of the normalized price is given by

$$R^{BG} p_t^{BG} = (d^{BG} + p_{t+1}^{BG})\left(\frac{n_t^{BG}}{n_{t+1}^{BG}}\frac{w_{H,t+1}^{BG}}{w_{H,t}^{BG}}\right), \tag{3.32}$$

where R^{BG} is the rate of return of stocks and $d^{BG} := [(1-\gamma)/\gamma](1-\tau^{BG})H$.

Also, skilled labor's wage rate on the BGP is expressed as (see Appendix A.2

[11] The values of the variables in the state BG depend on when the economy transitions to the state BG but in (3.31), readers will be able to infer from assumption 1 that when $s_t = UG$, the economy transitions to the state BG at date $t+1$ with probability $1-\pi$. In this paper, we follow the convention used in Weil (1987) and many subsequent studies on stochastic bubbles.

for the derivation)

$$w_{H,t}^{BG} = \gamma(1-\alpha)(n_t^{BG})^{\phi^{BG}} \left[\alpha \left(\frac{L}{(1-\tau^{BG})H}\right)^{1-\rho} + 1 - \alpha\right]^{\frac{\rho}{1-\rho}}. \tag{3.33}$$

Substituting this into (3.32) yields

$$R^{BG} p_t^{BG} = (d^{BG} + p_{t+1}^{BG})(G_n^{BG})^{\phi^{BG}-1}, \tag{3.34}$$

where

$$G_n^{BG} := \frac{n_{t+1}^{BG}}{n_t^{BG}} = 1 + a\tau^{BG} H. \tag{3.35}$$

Solving (3.34) yields the normalized fundamental value of stocks

$$v^{BG} := \frac{d^{BG}}{[R^{BG}/(G_n^{BG})^{\phi^{BG}-1}] - 1}. \tag{3.36}$$

Substituting $p_t^{BG} = 1/a$ into (3.34) yields $R^{BG} = a(d^{BG} + 1/a)(G_n^{BG})^{\phi^{BG}-1}$. Then, substituting this R^{BG} into (3.36) yields $v^{BG} = 1/a$, i.e., $p_t^{BG} = v^{BG} \iff P_t^{BG} = V_t^{BG}$ for all $t \geq T$.

We will now consider the economy at time $t < T$ when $s_t = UG$ at time t. We focus on the asymptotic behavior of the model when the state of UG persists, i.e., $T \to \infty$. The reason we focus on asymptotic behavior is that whether stock bubbles emerge at present depends on whether they are expected to arise in the future, which in turn depends on the asymptotic behavior.

Under the assumption 5, asymptotically, the no-arbitrage condition (3.31) in terms of the normalized price will become (see Appendix A.2 for the derivation)

$$R^{UG} p_t^{UG} \sim \pi p_{t+1}^{UG} (G_n^{UG})^{\psi^{UG}-1}, \tag{3.37}$$

where

$$R^{UG} := \pi (G_n^{UG})^{\psi^{UG}-1} \tag{3.38}$$

and

$$G_n^{UG} := \lim_{t\to\infty} G_{n,t}^{UG} = 1 + aH. \tag{3.39}$$

That is, the normalized price p_t^{UG} asymptotically follows the no-arbitrage condition for assets without dividends. Hence, the normalized fundamental value will asymptotically become zero, while the normalized equilibrium price satisfies $p_t^{UG} = 1/a > 0$ for all t. This implies that the equilibrium stock price will eventually exceed its fundamental value for sufficiently large t as long as the state of UG persists, i.e., $P_{T-1}^{UG} > V_{T-1}^{UG}$ as $T \to \infty$. Moreover, if equilibrium stock prices are expected to contain bubbles in the future, bubbles will be included even in the current price. In other words, the price of the stock contains a bubble and $P_t^{UG} > V_t^{UG}$ for all $t \in [0, T)$.

It should be noted that the normalized fundamental value will asymptotically become zero but the non-normalized value, i.e., V_t^{UG}, is always positive because after the state of BG arises, $P_t^{BG} = V_t^{BG}$.

Implications for the price-dividend ratio We can also show that the dynamics of the price-dividend ratio is markedly different before and after T.

Before T, i.e., $t \in [0, T)$, since the normalized dividend d_{t+1}^{UG} in the no-arbitrage condition (3.31) asymptotically converges to zero as $T \to \infty$, this implies that the price-dividend ratio

$$\frac{P_t^{UG}}{D_t^{UG}} = \frac{p_t^{UG}}{d_t^{UG}} = \frac{(1/a)}{d_t^{UG}} \tag{3.40}$$

will rise divergently as long as the state of UG persists.

On the other hand, once the state of UG ends and the stock bubble collapses at time T, the price-dividend ratio

$$\frac{P_t^{BG}}{D_t^{BG}} = \frac{p_t^{BG}}{d_t^{BG}} = \frac{(1/a)}{d^{BG}} \tag{3.41}$$

gets constant after $t \geq T$.

(3.40) and (3.41) tell us that together with a regime change depending on UG and BG, the price-dividend ratio will initially rise with bubbles, which appears to be explosive, and then fall with their collapse. This dynamics of the price-rent ratio has a potential for connecting our analysis with the bubble detection literature (Phillips, Shi, and Yu, 2015; Phillips and Shi, 2018, 2020), which detects a bubble by an explosive dynamics in the price-dividend ratio. We should note that for the emergence of stochastic bubbles, the conditions (b) and (c) we have identified in Section 2 are also important as well as a divergent increase in the ratio.

We summarize these insights in the following Proposition.

Proposition 1 (Emergence of stochastic stock bubbles)**.** *Suppose that assumption* $(\phi^{UG} - \psi^{UG})(\rho - 1) > 0$ *and assumption 5. Additionally, assume that the initial condition* $f(0, n_0^{UG}) > 1/(aH)$ *holds. Then,* *(i) for all $t \in [0,T)$, the stock price P_t^{UG} contains a bubble, with a divergent increase in the price-dividend ratio, where T is the period in which the economy draws the probability $1-\pi$.*

(ii) After $t \geq T$, the stock price P_t^{BG} is equal to its fundamental value and the price-dividend ratio gets constant.

From Proposition 1, we learn that the dynamics with unbalanced growth and stock bubbles can be seen as a temporary deviation from the balanced growth path.

Intuitively, when $\psi^{UG} > \phi^{BG} = \psi^{BG}$, i.e., the spillover effects of innovation are high, and $\phi^{UG} \neq \psi^{UG}$, the positive effects of innovation are unevenly spread on the two production factors $A_t^{UG}L$, and $Z_t^{UG}X_t^{UG}$. This causes a divergence in their productivity growth rates, leading to unbalanced growth. This unbalanced growth, in turn, leads to a stock price bubble in industries that drive innovation. This implies that when $\rho > 1$, i.e., the elasticity of substitution between the two factors of production is less than one, bubbles are attached to the price of intangible capital with higher rates of growth. However, once the state of UG with high spillovers ends and the spillover effects of innovation are evenly spread across the two factors of production, the economy transitions to the BGP, and the stock price bubble bursts. Our model suggests that stock bubbles emerge in the process of spillover of technological innovation, which is consistent with one of the stylized facts highlighted by Scheinkman (2014, p. 22), as noted in the introduction.

It should be mentioned that we can verify that the economic conditions under which stochastic stock bubbles emerge in Proposition 1 are consistent with conditions (a), (b) and (c) that we have identified in Section 2.5. The correspondence can be explained as follows. $(\phi^{UG}-\psi^{UG})(\rho-1) > 0$ ensures the condition (a) in Section 2.5. Under $(\phi^{UG}-\psi^{UG})(\rho-1) > 0$, d_{t+1}^{UG} in the no-arbitrage condition (3.31) converges to zero, implying that P_t^{UG} grows faster than D_t^{UG}. On the other hand, $\psi^{UG} > \phi^{BG} = \psi^{BG}$ ensures conditions (b) and (c). Under $\psi^{UG} > \phi^{BG} = \psi^{BG}$, the second term involving $1-\pi$ in the no-arbitrage condition (3.31) converges to zero, implying that P_t^{UG} grows faster than both D_{t+1}^{BG} and P_{t+1}^{BG}.

Growth rate of $n_t^{s_t}$ before and after T We can further derive the dynamics of $G_{n,t}^{s_t}$ before and after T. Additionally suppose $(n_0^{UG})^{(\phi^{UG}-\psi^{UG})(\rho-1)} > 1$. Then,

$\tau_t^{UG} > \tau^{BG}$ holds for all $t \in [0, T)$. Therefore, we obtain

$$G_{n,t}^{UG} > G_{n,t}^{BG}. \tag{3.42}$$

This implies that the collapse of the stock bubbles at time T leads to lower growth in the number of varieties and less innovation after $t \geq T$.[12]

Intuitively, as long as the state of UG persists, the level of $R\&D$ activity τ_t^{UG} increases monotonically, generating more variety and enhancing innovations. Once the state of UG ends and the stock bubbles burst, the level of $R\&D$ activity τ_t^{UG} declines, leading to reduced growth of variety and innovations. Note that if $(n_0^{UG})^{(\phi^{UG}-\psi^{UG})(\rho-1)} \leq 1$, (3.42) holds for sufficiently large t when the state of UG persists.

We summarize these insights in the following Proposition.

Proposition 2 (Impact of stock bubbles on innovations). *Suppose that* $(\phi^{UG} - \psi^{UG})(\rho - 1) > 0$ *and assumption 5. Additionally, assume that the initial condition* $f(0, n_0^{UG}) > 1/(aH)$ *and* $(n_0^{UG})^{(\phi^{UG}-\psi^{UG})(\rho-1)} > 1$ *hold. Then,*

(i) for all $t \in [0, T)$*, the R&D activity level* τ_t^{UG} *increases monotonically over time, generating more variety and boosting innovations, where T is the period in which the economy draws the probability* $1 - \pi$.

(ii) At date T*, the R&D activity level* τ^{BG} *declines, and for* $t > T$*, the growth rate of the number of varieties gets lower. Together with Proposition 1, after the burst of stock bubbles, the macroeconomy enters a period of sluggish innovation.*

Under the bubble equilibrium with unbalanced growth, innovation increases the wages, driving stock prices to rise rapidly compared to the BGP. This rise in stock prices stimulates the development of new technologies, accelerating innovation, which, in turn, further enhances future wages, stock prices, and innovation activities. This dynamics creates a virtuous cycle of rising stock prices and innovation.

3.11 Long-term effects of stock bubbles

We now examine the long-run effects of stock bubbles. We obtain the following Proposition.

[12]Note that if we focus on the asymptotic behavior of $T \to \infty$, we learn from (3.21) that the output growth rate rises and falls together with an increase and decrease in $G_{n,t}^{s_t}$. Hence, the growth rate of aggregate output also decreases together with the burst of stock bubbles.

Proposition 3 (Long-term positive effects of stock bubbles)**.** *Suppose that* $(\phi^{UG}-\psi^{UG})(\rho-1)>0$ *and assumption 5. Additionally, assume that the initial conditions* $f(0,n_0^{UG})>1/(aH)$ *and* $(n_0^{UG})^{(\phi^{UG}-\psi^{UG})(\rho-1)}>1$ *hold. Then, the longer the bubble period* $T-1$*, the higher the GDP level after the bubble collapses. In other words, stock bubbles leave positive effects on the macroeconomy even after the collapse.*

Proof. From (3.26), the condition $(n_0^{UG})^{(\phi^{UG}-\psi^{UG})(\rho-1)}>1$ ensures that $\tau_t^{UG}>\tau^{BG}$ holds for all $t\in[0,T)$. Consequently, as T increases, the accumulation of knowledge n_t^{UG} increases. According to (3.33) and (3.47), both $w_{H,t}^{BG}=P_t^{BG}an_t^{BG}$ and Y_t^{BG} increase with n_t^{BG}. Therefore, an increase in T leads to a higher level of GDP: $Y_t^{BG}+P_t^{BG}(n_{t+1}^{BG}-n_t^{BG})=Y_t^{BG}+w_{H,t}^{BG}\tau^{BG}H$ for all $t\geq T$. □

This proposition gives us an important insight into the light side of stock price bubbles. That is to say, during the bubble period, innovation is producing a wide variety of goods and technologies, and the economy is enjoying it. Even after the bubble collapses, the technologies n_t^{UG} developed during the bubble period will survive, resulting in a higher GDP level even after the collapse. This positive effect is stronger as the bubble period is longer. As noted in the introduction, this light side of stock bubbles is also consistent with the narrative highlighted by Scheinkman (2014), i.e., stock bubbles may have positive effects on innovative investments and economic growth.[13]

Short-term effects of the collapse of stock bubbles While stock bubbles leave long-term positive effects, their collapse leads to a greater decline in GDP and stock prices the longer the bubble lasts. To see this point, let us focus on the asymptotic behavior for analytical tractability.

Let us first investigate the collapse of stock bubbles on the dynamics of the stock price $P_t^{s_t}$. From (3.30), the stock price is given by

$$P_t^{s_t}=\left(\frac{1}{a}\right)\left(\frac{w_{H,t}^{s_t}}{n_t^{s_t}}\right). \tag{3.43}$$

Considering the situation where $s_t=UG$ but $s_{t+1}=BG$, (3.43) leads to

$$P_t^{UG}>P_{t+1}^{BG}\iff\frac{n_{t+1}^{BG}}{n_t^{UG}}>\frac{w_{H,t+1}^{BG}}{w_{H,t}^{UG}}. \tag{3.44}$$

[13] Scheinkman (2014) also discusses that instead of stock bubbles, the consequences of leveraged real estate bubbles can be different because they can lead to devastating consequences in the financial system.

When $(\phi^{UG} - \psi^{UG})(\rho - 1) > 0$, $n_{t+1}^{BG}/n_t^{UG} \sim G_n^{UG}$ holds asymptotically. The intuition is that if the state of UG persists, the level of R&D activity $\tau_t^{UG} \to 1$ holds. Even if the state transitions to BG in the following period, the value of the state variable n_{t+1}^{BG} is already determined at date t. Therefore, $n_{t+1}^{BG}/n_t^{UG} \sim G_n^{UG}$ holds. In addition, under the assumption 5, $w_{H,t+1}^{BG}/w_{H,t}^{UG} \sim 0$ (see Appendix A.2 for the derivation). Intuitively, the wage for skilled labor $w_{H,t}^{s_t}$ grows at the same rate as $Y_t^{s_t}$ asymptotically. From (3.46) below, the growth rate of output Y_t^{UG} in the state UG is asymptotically $(G_n^{UG})^{\psi^{UG}}$. On the other hand, from (3.47) below, and $n_{t+1}^{BG}/n_t^{UG} \sim G_n^{UG}$, the growth rate of the hypothetical output Y_{t+1}^{BG} in state BG conditional on the state being UG at date t is $(G_n^{UG})^{\phi^{BG}}$. Therefore, $w_{H,t+1}^{BG}/w_{H,t}^{UG} \sim 0$ holds, i.e., the skilled labor wage in the state UG grows faster than that on the hypothetical BGP. For these two reasons, inequality (3.44) eventually holds, indicating that the collapse of the bubble leads to a sharper decline in stock prices when the bubble period is longer. This insight is the same as the insight we have derived in the condition (b) we have identified in Section 2.5.

Moreover, we can also derive another insight into the stagnation of the stock price after the burst of stock bubbles. As explained in the paragraph above, since the skilled labor wage $w_{H,t}^{s_t}$ grows at the same rate as output $Y_t^{s_t}$ asymptotically, we have $w_{H,t+1}^{UG}/w_{H,t}^{UG} \sim (G_n^{UG})^{\psi^{UG}}$ and $w_{H,t+1}^{BG}/w_{H,t}^{BG} = (G_n^{BG})^{\phi^{BG}}$ (see Appendix A.2 for the derivation). Thus, since $\psi^{UG} > \phi^{BG} = \psi^{BG}$ and $G_n^{UG} > G_n^{BG} \geq 1$, from (3.43), the following inequality holds asymptotically between any date t and $t+1$:

$$\frac{P_{t+1}^{UG}}{P_t^{UG}} \sim (G_n^{UG})^{\psi^{UG}-1} > \frac{P_{t+1}^{BG}}{P_t^{BG}} = (G_n^{BG})^{\phi^{BG}-1}, \tag{3.45}$$

indicating that the growth rate of stock prices decreases after the burst of stock bubbles. As noted already, the economy transitions to the BGP immediately without transitional dynamics. Hence, the right-hand side of (3.45) holds immediately after the economy moves to the state of BG.

Next, we examine the effects of the collapse of stock bubbles on the dynamics of $Y_t^{s_t}$. Under the condition $(\phi^{UG} - \psi^{UG})(\rho - 1) > 0$, $\lim_{t\to\infty} Y_t^{UG}/(A_t^{UG}L) = \alpha^{1/(1-\rho)}$ holds. Therefore, the output Y_t^{UG} in the state of UG asymptotically becomes

$$Y_t^{UG} \sim \alpha^{\frac{1}{1-\rho}} A_t^{UG} L = \alpha^{\frac{1}{1-\rho}} (n_t^{UG})^{\psi^{UG}} L. \tag{3.46}$$

The intuition is as follows. Suppose $\phi^{UG} > \psi^{UG}$ and $\rho > 1$. This means that the two factors of production, i.e., $Z_t^{UG} X_t^{UG}$ and $A_t^{UG} L$, are complementary, and the former is growing faster than the latter. In this case, the production factor with slower growth $A_t^{UG}L$ asymptotically determines the output level Y_t^{UG}. On the other hand, suppose that $\phi^{UG} < \psi^{UG}$ and $\rho < 1$. This means that the productivity growth of $A_t^{UG}L$ is faster than that of $Z_t^{UG}X_t^{UG}$, and the two factors of production are substitutional. In this case, the production factor with faster growth $A_t^{UG}L$ asymptotically determines the output level Y_t^{UG}.

Once the macroeconomy falls into the state of BG, from (3.21), the output Y_t^{BG} is given by

$$Y_t^{BG} = (n_t^{BG})^{\phi^{BG}} \left[\alpha L^{1-\rho} + (1-\alpha)[(1-\tau^{BG})H]^{1-\rho}\right]^{\frac{1}{1-\rho}}. \tag{3.47}$$

Then, from (3.46) and (3.47), at any date between t and $t+1$, we asymptotically obtain the inequality

$$\frac{Y_{t+1}^{UG}}{Y_t^{UG}} \sim (G_n^{UG})^{\psi^{UG}} > \frac{Y_{t+1}^{BG}}{Y_t^{BG}} = (G_n^{BG})^{\phi^{BG}}. \tag{3.48}$$

Thus, the growth of output stagnates after the bubble bursts.

Moreover, consider the situation where $s_t = UG$ but $s_{t+1} = BG$. Then, since $n_{t+1}^{BG}/n_t^{UG} \sim G_n^{UG}$ asymptotically and the hypothetical output Y_{t+1}^{BG} in the state BG conditional on the state being UG at date t increases at the rate of $(G_n^{UG})^{\phi^{BG}}$ asymptotically, under the assumption 5, the following inequality eventually holds:

$$Y_t^{UG} \sim \alpha^{\frac{1}{1-\rho}} (n_t^{UG})^{\psi^{UG}} L > Y_{t+1}^{BG}. \tag{3.49}$$

Consequently, the collapse of the stock bubble not only causes a sharper decline in stock prices, but also generates a sharper decline in the output level $Y_t^{s_t}$ when the bubble period is longer.

In summary, the longer the bubble period, the more the collapse of the bubble significantly reduces the output $Y_t^{s_t}$, stock price $P_t^{s_t}$, and R&D activities $\tau_t^{s_t}$. Consequently, GDP $Y_t^{s_t} + P_t^{s_t}(n_{t+1}^{s_{t+1}} - n_t^{s_t})$ also experiences a sharper decline.

4 Discussions

4.1 Relation to Uzawa steady-state growth theorem

The model we have presented implies that the emergence of stock price bubbles is closely related to Uzawa steady-state growth theorem (Uzawa, 1961), which is at the heart of macro-growth theory. Typically, the Uzawa's theorem is analyzed within the context of neoclassical growth theory to explore the characteristics of technological progress necessary for achieving a BGP. As is well known, the aggregate production function must exhibit Harrod-neutral (purely labor-augmenting) technological progress on the BGP. Here, we extend the application of the Uzawa's theorem to our framework for discussion.

Proposition 4. *(Uzawa steady-state growth theorem) Consider the aggregate production function of consumption goods (3.1) where $Z_t^{s_t}$ and $A_t^{s_t}$ are given by (3.19) and (3.20), respectively. The aggregate production of the good $X_t^{s_t}$ satisfies (3.9), i.e., all intermediate goods $x_t^{s_t}(i)$ has the same production level $x_t^{s_t}$. The evolution of number of varieties is given by (3.15). The labor market clearing condition (3.16) is satisfied. Suppose there are constant growth rates such that $Y_{t+1}^{s_{t+1}}/Y_t^{s_t} = G_Y > 0$ and $n_{t+1}^{s_{t+1}}/n_t^{s_t} = G_n > 0$ for all $t \geq t_0$, i.e., the economy is on a BGP. Then, $(\phi^{s_t} - \psi^{s_t})(\rho - 1) = 0$ must hold for all $t \geq t_0$.*

Proof. See Appendix A.3. □

In the case of a Cobb-Douglas production function ($\rho = 1$), which is a knife-edge case, a BGP can be achieved even if there is a difference in the growth rates of the two production factors $A_t^{s_t} L$ and $Z_t^{s_t} X_t^{s_t}$, i.e., $\phi^{s_t} \neq \psi^{s_t}$ is acceptable for a BGP. However, if the production function is not Cobb-Douglas ($\rho \neq 1$), the growth rates of the two factors must be equal to achieve a BGP, i.e., $\phi^{s_t} = \psi^{s_t}$ must hold. In either case, we can see that the parameters for ensuring balanced growth are knife-edge.[14] Based on Propositions 1 and 4, the relationship between the emergence of stock price bubbles and Uzawa's theorem becomes clear in the following corollary.

Corollary 1. *The bubbly equilibrium in Proposition 1 fails to satisfy the Uzawa steady-state growth theorem, i.e., $(\phi^{UG} - \psi^{UG})(\rho - 1) = 0$ does not hold.*

What we can learn from Propositions 1, 4, and Corollary 1 is that (i) even the slightest deviation from the condition $(\phi^{s_t} - \psi^{s_t})(\rho - 1) = 0$ leads to a world of

[14] In Uzawa's theorem within neoclassical growth theory, capital-augmenting technological progress must be entirely absent on the BGP, which represents a knife-edge assumption. Notably, when the aggregate production function is Cobb-Douglas, capital-augmenting technological progress can be incorporated by reinterpreting it as labor-augmenting technological progress. Furthermore, the assumption that the elasticity of substitution between capital and labor is exactly 1 also constitutes a knife-edge condition. Acemoglu (2009, §15.6) constructs a model of directed technological change, where the productivity growth of two production factors is determined by two sectors of R&D investment. He shows that balanced growth can only be achieved under a knife-edge condition. Grossman et al. (2017) consider a general neoclassical production function and obtain balanced growth, despite the presence of capital-augmenting technological progress. Nevertheless, it still requires knife-edge conditions. Indeed, Grossman, Helpman, Oberfield, and Sampson (2017, p. 1306) clearly note “our model is no exception to this rule.”

unbalanced growth, where different production factors grow at different rates, and (ii) in the world of unbalanced growth, the condition $(\phi^{s_t} - \psi^{s_t})(\rho - 1) > 0$, even if it arises only temporarily, leads to the emergence of asset price bubbles.

4.2 Related literature

Our paper belongs to the so-called "rational bubble literature" that studies bubbles as speculation backed by nothing, which was pioneered by Samuelson (1958), Bewley (1980), Tirole (1985), Scheinkman and Weiss (1986), Kocherlakota (1992), and Santos and Woodford (1997). Theoretical foundations in rational bubble models include Huang and Werner (2000), Hellwig and Lorenzoni (2009), and Bloise and Citanna (2019), among others.

As noted in the introduction, it is well known in the literature on rational bubbles that there is a fundamental difficulty in generating bubbles in real assets with positive dividends. This difficulty follows from Santos and Woodford (1997, Theorem 3.3, Corollary 3.4), who show that, when the asset pays non-negligible dividends relative to the aggregate endowment, bubbles are impossible.[15] This "Bubble Impossibility Theorem" has been extended under alternative financial constraints by Kocherlakota (2008) and Werner (2014).

Due to the fundamental difficulty,[16] there are only a handful of papers that deal with bubbles attached to real assets, including Wilson (1981, §7), Tirole (1985, Proposition 1(c)),[17] Olivier (2000), and Bosi et al. (2018).[18] A series of papers by (Hirano and Toda, 2023a, 2024, 2025) and Hirano, Jinnai, and Toda (2022)

[15] This difficulty also implies that there is a discontinuity in proving the existence of a bubble between zero-dividend assets (pure bubble assets like fiat money or cryptocurrency) and dividend-paying assets such as stocks, land, and housing.

[16] Because of the difficulty, the rational bubble literature has almost exclusively focused on pure bubbles without dividends. Since the literature on pure bubbles is too large, we would like readers to see Hirano and Toda (2024) for a recent review of pure-bubble models and criticisms from the general audience to those models. See also Barlevy (2018), who discusses that pure bubble models face fundamental limitations for applications including policy and quantitative analyses. Importantly, as Hirano and Toda (2025) show, the economic insights and implications are markedly different between pure bubbles and bubbles attached to real assets.

[17] See Hirano and Toda (2024, §5.2) and Hirano and Toda (2025, §V.A) for the differences of their series of papers from Wilson (1981, §7), and Tirole (1985, Proposition 1(c)).

[18] Bosi et al. (2018) study Tirole (1985)'s model in the presence of altruism. Olivier (2000) examines deterministic bubbles attached to individual stocks in a world in which the law of one price is violated, i.e., even if the dividend and the fundamental value of stocks are the same, the price of each stock can be different. In contrast, in our model of stock prices in Section 3, we assume the law of one price, as in ordinary macro-finance models, i.e., if the fundamental value of stocks is the same, the price is the same. In Bosi et al. (2018) and Olivier (2000), there is a continuum of bubble equilibria as in pure bubble models. In contrast, in the present paper and a series of papers by (Hirano and Toda, 2023a, 2024, 2025) and Hirano, Jinnai, and Toda (2022), equilibrium is uniquely determined.

not only show bubbles attached to real assets within workhorse macroeconomic models, but also present a conceptually new perspective of the necessity of asset price bubbles, as noted in the introduction. The present paper advances the direction their series of papers have opened up and considers stochastic bubbles that are expected to collapse, while all papers noted above study deterministic bubbles that are not expected to collapse.

Regarding the point that we consider aggregate risk, the present paper is most closely related to Hirano and Toda (2023b), who study the implications for land prices in economies with aggregate uncertainty. They show that land prices exhibit recurrent stochastic fluctuations, with expansions and contractions in the size of land bubbles. A critical difference of our paper from Hirano and Toda (2023b) is that we consider bubbles that are expected to collapse completely, while in their paper, land prices always contain a bubble and the size of bubbles changes all the time.

4.3 Concluding remarks

As noted in the introduction, any balanced growth model is knife-edge theory. By imposing knife-edge restrictions, macro-models that generate a BGP are constructed from the beginning so that asset prices are equal to the fundamentals. We have shown that the slightest deviation from knife-edge cases leads to markedly different asset pricing implications. To illustrate this point, as an example of a full-fledged macro-finance model, we have employed the innovation-driven growth model proposed by Grossman and Helpman (1991a, Ch.3). Our approach that allows for the possibility of unbalanced growth and considers regime switching with unbalanced growth and balanced growth can generally be applied to other modern innovation-driven growth models, including Romer (1990), Grossman and Helpman (1991b), and Aghion and Howitt (1992).

Similarly, in many cases, by imposing knife-edge conditions, macro-finance models are constructed from the outset so that the macroeconomy converges to a steady state characterized by balanced growth. In other words, the macroeconomy is always on the same dynamic path, which usually corresponds to a saddle path that can be drawn with one stroke of the brush. By adding various types of exogenous shocks or by changing the magnitude of those shocks, macro-finance models have attempted to account for fluctuations in asset prices qualitatively and quantitatively, which has produced fruitful outcomes up to present. In light of this existing approach, it would be fair to say that our methodology of macro-finance-theory construction would provide a different approach. That is, our approach of removing the knife-edge restrictions allows the macroeconomy to temporarily take a different dynamic path from a balanced growth path. As our paper has illustrated, this deviation from the BGP would result in markedly different implications for asset pricing. In other words, the dynamic path with asset price bubbles can be understood as a temporary deviation from the BGP. Also, in the present paper, to illustrate the key conditions and mechanisms of stochastic bubbles, we abstract from financial frictions and financial accelerator effects, which play an important role in recent macro-finance models, including influential papers such as Greenwald et al. (1984), Bernanke and Gertler (1989), Greenwald and Stiglitz (1993), Kiyotaki and Moore (1997), Krishnamurthy (2003), Lorenzoni (2008), Brunnermeier and Sannikov (2014), etc. Our approach can be embedded into these representative models of financial accelerator. As such, we hope that our construction of a macro-finance-model where unbalanced growth dynamics can temporarily occur would provide a new direction in macro-finance theory.

A Appendix

A.1 Derivation of (3.26)

According to (3.15), (3.17) becomes

$$1-\tau_t^{s_t}+\frac{w_{L,t}^{s_t}L}{w_{H,t}^{s_t}H}=\frac{n_t^{s_t}P_t^{s_t}}{w_{H,t}^{s_t}H}. \tag{A.1}$$

From (3.8), (3.6) becomes

$$Q_t^{s_t}=\frac{w_{H,t}^{s_t}}{\gamma}(n_t^{s_t})^{-\frac{1-\gamma}{\gamma}}. \tag{A.2}$$

Then, according to (3.3) and (A.2), we obtain

$$w_{H,t}^{s_t}=(1-\alpha)\gamma(n_t^{s_t})^{\frac{1-\gamma}{\gamma}}Z_t^{s_t}\left(\frac{Y_t^{s_t}}{Z_t^{s_t}X_t^{s_t}}\right)^{\rho}. \tag{A.3}$$

From (3.2), (3.18), (3.19), (3.20), and (A.3), we obtain

$$\frac{w_{L,t}^{s_t}}{w_{H,t}^{s_t}}=\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\gamma}\right)\left(\frac{(n_t^{s_t})^{\phi^{s_t}}}{(n_t^{s_t})^{\psi^{s_t}}}\right)^{\rho-1}\left(\frac{(1-\tau_t^{s_t})H}{L}\right)^{\rho}. \tag{A.4}$$

(3.14) is rewritten as

$$\frac{n_t^{s_t} P_t^{s_t}}{w_{H,t}^{s_t}} = \frac{1}{a}. \tag{A.5}$$

Substituting (A.4) and (A.5) into (A.1) yields (3.26).

A.2 The asymptotic behavior of each variable in the case of $(\phi^{UG} - \psi^{UG})(\rho - 1) > 0$

First, we investigate the asymptotic behaviors of τ_t^{UG}, Y_t^{UG}, and $w_{H,t}^{UG}$, respectively, when the state of $s_t = UG$ persists. Suppose $(\phi^{UG} - \psi^{UG})(\rho - 1) > 0$ and $f(0, n_0^{UG}) > 1/(aH)$. Then, since $\lim_{t\to\infty} \tau_t^{UG} = 1$, $f(\tau_t^{UG}, n_t^{UG}) = 1/(aH)$ for all t, and $\lim_{t\to\infty} n_{t+1}^{UG}/n_t^{UG} = 1 + aH =: G_n^{UG}$, the second term in (3.27) must converge to a positive constant. Thus, asymptotically, the following relationship holds:

$$1 - \tau_t^{UG} \sim (1 - \tau^{UG})(G_n^{UG})^{-(\phi^{UG} - \psi^{UG})[(\rho-1)/\rho]t}, \tag{A.6}$$

where $1 - \tau^{UG} > 0$ is a positive constant.

Furthermore, since $\lim_{t\to\infty} Y_t^{UG}/[(n_t^{UG})^{\psi^{UG}} L] = \alpha^{1/(1-\rho)}$ holds,

$$Y_t^{UG} \sim \alpha^{\frac{1}{1-\rho}} (n_t^{UG})^{\psi^{UG}} L. \tag{A.7}$$

From (3.3) and (A.2), we obtain

$$w_{H,t}^{UG} = \gamma(1-\alpha)(n_t^{UG})^{\phi^{UG}(1-\rho)} \left(\frac{Y_t^{UG}}{(1 - \tau_t^{UG})H} \right)^{\rho}. \tag{A.8}$$

Then, substituting (A.6) and (A.7) into (A.8) obtains

$$w_{H,t}^{UG} \sim w_H^{UG} (G_n^{UG})^{\psi^{UG} t}, \tag{A.9}$$

where $w_H^{UG} > 0$ is a constant.

Next, derive Q_t^{BG} and $w_{H,t}^{BG}$. From (3.6), (3.18), (3.19), (3.21), and using the condition $\phi^{BG} = \psi^{BG}$, we can derive Q_t^{BG} as follows:

$$Q_t^{BG} = (1-\alpha) Z_t^{BG} \left[\alpha \left(\frac{L}{(1 - \tau^{BG})H} \right)^{1-\rho} + 1 - \alpha \right]^{\frac{\rho}{1-\rho}}, \tag{A.10}$$

where τ^{BG} satisfies (3.28). From (A.2) and (A.10), we derive $w_{H,t}^{BG}$ as follows:

$$w_{H,t}^{BG} = \gamma(1-\alpha)(n_t^{BG})^{\phi^{BG}} \left[\alpha \left(\frac{L}{(1-\tau^{BG})H} \right)^{1-\rho} + 1 - \alpha \right]^{\frac{\rho}{1-\rho}}. \tag{A.11}$$

Consider the asymptotic behavior of the no-arbitrage condition (3.31) when the state of $s = UG$ persists. $n_{t+1}^{UG} \sim n_t^{UG} G_n^{UG}$ holds. Since the hypothetical value of n_{t+1}^{BG} conditional on the state being UG at date t is determined in the period t, $n_{t+1}^{BG} \sim n_t^{UG} G_n^{UG}$ also holds. Then, from (A.9) and (A.11), we obtain

$$\frac{w_{H,t+1}^{BG}}{w_{H,t}^{UG}} \sim \frac{\gamma(1-\alpha)(n_t^{UG} G_n^{UG})^{\phi^{BG}} \left[\alpha \left(\frac{L}{(1-\tau^{BG})H} \right)^{1-\rho} + 1 - \alpha \right]^{\frac{\rho}{1-\rho}}}{w_H^{UG} (G_n^{UG})^{\psi^{UG} t}}. \tag{A.12}$$

Since n_t^{UG} grows at rate G_n^{UG},

$$\lim_{t \to \infty} \frac{w_{H,t+1}^{BG}}{w_{H,t}^{UG}} = \begin{cases} 0 & \text{if } \psi^{UG} > \phi^{BG} = \psi^{BG} \\ \infty & \text{if } \phi^{BG} = \psi^{BG} > \psi^{UG} \end{cases} \tag{A.13}$$

holds.

Derive the asymptotic no-arbitrage condition when $\psi^{UG} > \phi^{BG}$. Noting that $d_t^{UG} = [(1-\gamma)/\gamma](1-\tau_t^{UG})H \to 0$, d_t^{BG} is constant, and $p_t^{st} = 1/a$, the asymptotic rate of return is derived from (3.31) as:

$$\begin{aligned} R_t^{UG} &= \pi \left(\underbrace{\frac{d_{t+1}^{UG}}{p_t^{UG}}}_{0} + \underbrace{\frac{p_{t+1}^{UG}}{p_t^{UG}}}_{1} \right) \left(\underbrace{\frac{n_t^{UG}}{n_{t+1}^{UG}}}_{1/G_n^{UG}} \underbrace{\frac{w_{H,t+1}^{UG}}{w_{H,t}^{UG}}}_{(G_n^{UG})^{\psi^{UG}}} \right) + (1-\pi) \left(\frac{d_{t+1}^{BG}}{p_t^{UG}} + \frac{p_{t+1}^{BG}}{p_t^{UG}} \right) \underbrace{\left(\frac{n_t^{UG}}{n_{t+1}^{BG}} \frac{w_{H,t+1}^{BG}}{w_{H,t}^{UG}} \right)}_{0} \\ &\sim \pi (G_n^{UG})^{\psi^{UG}-1} =: R^{UG}. \end{aligned} \tag{A.14}$$

From this, the asymptotic no-arbitrage condition is

$$\begin{aligned} R^{UG} p_t^{UG} &= \pi \left(\underbrace{d_{t+1}^{UG}}_{0} + p_{t+1}^{UG} \right) \left(\underbrace{\frac{n_t^{UG}}{n_{t+1}^{UG}}}_{1/G_n^{UG}} \underbrace{\frac{w_{H,t+1}^{UG}}{w_{H,t}^{UG}}}_{(G_n^{UG})^{\psi^{UG}}} \right) + (1-\pi) \left(d_{t+1}^{BG} + p_{t+1}^{BG} \right) \underbrace{\left(\frac{n_t^{UG}}{n_{t+1}^{BG}} \frac{w_{H,t+1}^{BG}}{w_{H,t}^{UG}} \right)}_{0} \\ &\sim \pi p_{t+1}^{UG} (G_n^{UG})^{\psi^{UG}-1}. \end{aligned} \tag{A.15}$$

A.3 Proof of Proposition 4

From (3.15), $\tau_t^{s_t}$ must remain constant for all $t \geq t_0$ to ensure a constant growth of $n_{t+1}^{s_{t+1}}/n_t^{s_t} = G_n$. Suppose $\tau_t^{s_t} = \tau^{s_t}$ for all $t \geq t_0$, where $\tau^{s_t} \in (0,1)$ is a constant. From (3.9), (3.16), (3.19), and (3.20), the production function of consumption good (3.1) is rewritten by (3.21), where $\tau_t^{s_t} = \tau^{s_t}$, i.e.,

$$Y_t^{s_t} = \left[\alpha[(n_t^{s_t})^{\psi^{s_t}} L]^{1-\rho} + (1-\alpha)[(n_t^{s_t})^{\phi^{s_t}}(1-\tau^{s_t})H]^{1-\rho}\right]^{\frac{1}{1-\rho}}. \tag{A.16}$$

Since $Y_t^{s_t} = Y_{t_0}^{s_{t_0}} G_Y^{t-t_0}$ and $n_t^{s_t} = n_{t_0}^{s_{t_0}} G_n^{t-t_0}$, the production function (A.16) at date t_0 is

$$Y_{t_0}^{s_0} = \left\{ \alpha[(\underbrace{n_t^{s_t} G_n^{-(t-t_0)}}_{n_{t_0}^{s_{t_0}}})^{\psi^{s_t}} L]^{1-\rho} + (1-\alpha)[(\underbrace{n_t^{s_t} G_n^{-(t-t_0)}}_{n_{t_0}^{s_{t_0}}})^{\phi^{s_t}}(1-\tau^{s_t})H]^{1-\rho} \right\}^{\frac{1}{1-\rho}}. \tag{A.17}$$

Multiplying both sides of (A.17) by $G_Y^{t-t_0}$ yields

$$Y_t^{s_t} = \left[\alpha\left(\frac{G_Y^{t-t_0}}{G_n^{\psi^{s_t}(t-t_0)}}(n_t^{s_t})^{\psi^{s_t}} L\right)^{1-\rho} + (1-\alpha)\left(\frac{G_Y^{t-t_0}}{G_n^{\phi^{s_t}(t-t_0)}}(n_t^{s_t})^{\phi^{s_t}}(1-\tau^{s_t})H\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}. \tag{A.18}$$

Comparing (A.16) and (A.18), $G_Y^{s_t} = G_n^{\phi^{s_t}} = G_n^{\psi^{s_t}}$ must hold for $\rho \neq 1$. This implies $\phi^{s_t} = \psi^{s_t}$.

Next consider the case of $\rho = 1$ (Cobb-Douglas production function), i.e., (A.16) becomes

$$Y_t^{s_t} = [(n_t^{s_t})^{\psi^{s_t}} L]^{\alpha}[(n_t^{s_t})^{\phi^{s_t}}(1-\tau^{s_t})H]^{1-\alpha}. \tag{A.19}$$

Repeat the same steps of the proof. At the date t_0, (A.19) becomes

$$Y_{t_0}^{s_0} = [(n_t^{s_t} G_n^{-(t-t_0)})^{\psi^{s_t}} L]^{\alpha}[(n_t^{s_t} G_n^{-(t-t_0)})^{\phi^{s_t}}(1-\tau^{s_t})H]^{1-\alpha}. \tag{A.20}$$

Multiplying both sides of (A.20) by $G_Y^{t-t_0}$ yields

$$Y_t^{s_t} = \left(\frac{G_Y}{G_n^{\psi^{s_t}}}\right)^{\alpha(t-t_0)} \left(\frac{G_Y}{G_n^{\phi^{s_t}}}\right)^{(1-\alpha)(t-t_0)} [(n_t^{s_t})^{\psi^{s_t}} L]^{\alpha}[(n_t^{s_t})^{\phi^{s_t}}(1-\tau^{s_t})H]^{1-\alpha}. \tag{A.21}$$

Comparing (A.19) and (A.21), $G_Y = G_n^{\alpha\psi^{s_t}+(1-\alpha)\phi^{s_t}}$ must hold for $\rho = 1$. This can hold even if $\phi^{s_t} \neq \psi^{s_t}$.

To summarize the above results, $(\phi^{s_t} - \psi^{s_t})(\rho - 1) = 0$ must hold on a BGP.

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