Housing Bubbles with Phase Transitions

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Abstract

We analyze equilibrium housing prices in an overlapping generations model with perfect housing and rental markets. The economy exhibits a two-stage phase transition: as the income of home buyers rises, the equilibrium regime changes from fundamental to bubble possibility, where fundamental and bubbly equilibria can coexist. With even higher incomes, fundamental equilibria disappear and housing bubbles become a necessity. Even with low current incomes, housing bubbles may emerge if home buyers have access to credit or have high future income expectations. Contrary to widely-held beliefs, fundamental equilibria in the possibility regime are inefficient despite housing being a productive non-reproducible asset.

Keywords: bubble, expectations, housing, phase transition, welfare.

JEL codes: D53, G12, R21.

1 Introduction

Over the last three decades, many countries have experienced appreciation in housing prices, with upward trends in the price-rent ratio.[1] The situation is often referred to in the popular press as a housing bubble. Because fluctuations in housing prices are often associated with macroeconomic problems, many academics and policymakers want to understand why and how housing bubbles emerge in the first place. However, the mechanism of the emergence of housing bubbles is poorly understood. In addition, theoretically, it is well known that there is a fundamental

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[1] See, for instance, Figure 1 of Amaral et al. (2024) for 27 major agglomerations in 15 OECD countries and U.S. Metropolitan Statistical Areas.

difficulty in generating asset price bubbles in dividend-paying assets such as housing, land, and stocks.[2] The theory of rational bubbles attached to assets with positive dividends remains largely underdeveloped: at present, there is no theoretical framework for considering whether housing prices reflect fundamentals or contain bubbles.

The primary purpose of this paper is to fill this gap and to present a theory of rational housing bubbles. In particular, we are interested in the following questions. (i) How can equilibrium housing prices be disconnected from fundamentals in the long term, exhibiting a bubble in a dynamic general equilibrium setting in which housing rents and prices are both endogenously determined? (ii) How is the disconnection related to economic conditions such as the income or access to credit of home buyers and to the formation of expectations about future economic conditions? (iii) What are the efficiency properties of equilibria?

To provide theoretical answers to these questions in the simplest possible setting, we consider a bare-bones model of housing. The economy is inhabited by overlapping generations that live for two periods (young and old age) and consume two commodities (consumption good and housing service). The ownership and occupancy of a house are separated, so there is a price for house ownership as a financial asset (housing price) and a price for house occupancy as a commodity (rent). The good, housing, and rental markets are all competitive and frictionless. A rational expectations equilibrium consists of a sequence of prices (housing price and rent) and allocations (consumption good, housing stock, and housing service) such that all agents optimize and markets clear. An equilibrium is *fundamental* (*bubbly*) if the housing price equals (exceeds) the present value of rents. In this model, the dividend of housing, namely rent, is endogenously determined by the demand and supply of housing. If housing supply is inelastic, as the economy grows and agents get richer or have access to more credit, they increase the demand for housing, which pushes up both the housing price and rent. Under these circumstances, it is not obvious whether housing prices will grow faster than rents and a housing bubble emerge: the possibility or necessity of housing bubbles becomes a nontrivial theoretical question.

We obtain three main results. First, we theoretically identify the economic conditions under which the equilibrium housing price reflects the fundamentals in the long run or exhibits a bubble. We prove that the economy experiences a

[2]Santos and Woodford (1997, Theorem 3.3, Corollary 3.4) show that, when the asset pays nonnegligible dividends relative to the aggregate endowment, bubbles are impossible. See the recent review article Hirano and Toda (2024a), especially §3.4, for a simple illustration.

two-stage phase transition. When the long run income ratio of the young (home buyers) relative to the old (home sellers) is sufficiently low, housing bubbles cannot arise and a fundamental equilibrium exists, which we refer to as the *fundamental regime*. When the ratio rises and exceeds the first critical value, a phase transition occurs.[3] Both a fundamental and a bubbly equilibrium exist, and the equilibrium is selected by agents' self-fulfilling expectations. We refer to this coexistence region as the *bubble possibility regime*. When the income ratio exceeds the second and still higher critical value, another phase transition takes place to the *bubble necessity regime*, where fundamental equilibria do not exist and housing bubbles become inevitable. Furthermore, we prove the uniqueness of equilibrium under weak conditions. We show that the fundamental equilibrium is always unique, and the bubbly equilibrium is unique if the elasticity of intertemporal substitution is not too much below 1/2.

The intuition for this two-stage phase transition is the following. Let $G > 1$ be the long run growth rate of the economy and $\gamma > 0$ the reciprocal of the elasticity of substitution between consumption and housing, which in the model also equals the elasticity of rent with respect to income. Empirical estimates suggest $\gamma < 1$,[4] and a theoretical argument also supports it: if $\gamma > 1$, as the economy grows and agents get richer, the young asymptotically spend all income on housing, the price-rent ratio converges to zero, and the interest rate diverges to infinity, which are all pathological and counterfactual. Since $\gamma = 1$ (Cobb-Douglas) is a knife-edge case, it is natural to focus on the case $\gamma < 1$. Under this condition, by equating marginal utility to prices, consumption grows at rate G but the rent grows at rate $G^\gamma < G$. Therefore, if the housing price only reflects fundamentals in the long run equilibrium, it must also grow at rate G^γ. Since housing price grows slower than endowments in any fundamental equilibrium, the expenditure share of housing converges to zero in the long run and the interest rate R is pinned down as the marginal rate of intertemporal substitution in the autarky allocation. If $R > G^\gamma$, a fundamental equilibrium exists. If $R < G^\gamma$, a fundamental equilibrium cannot exist, for otherwise the fundamental value of housing (the present value of

[3] *Phase transition* is a technical term in natural sciences that refers to a discontinuous change in the state as we change a parameter continuously, for instance the matter changes from solid to liquid to gas as we increase the temperature. The analogy here is appropriate because the regime of the economy abruptly changes from fundamental to bubbly as income rises.

[4] Ogaki and Reinhart (1998, Table 2) estimate the elasticity of substitution between durable and nondurable goods using aggregate data and obtain $\gamma = 1/1.24 = 0.81$. Piazzesi et al. (2007, Appendix C) estimate a cointegrating equation between the price and quantity of housing service relative to consumption using aggregate data and obtain $\gamma = 1/1.27 = 0.79$. Howard and Liebersohn (2021, Table 2) estimate $\gamma = 0.79$ using cross-sectional data on income and rents.

rents) becomes infinite, which is obviously impossible in equilibrium. Therefore as the young become richer and the interest rate falls below a certain threshold, the fundamental equilibrium becomes unsustainable, and a housing bubble inevitably emerges. In the long run equilibrium with housing bubbles we must have $R = G$ so that the bubble is just sustainable. Fundamental and bubbly equilibria coexist when the autarkic interest rate satisfies $G^\gamma < R < G$, which corresponds to an intermediate range for the income ratio of the young.

As our second main result, we show the possibility of credit- and expectation-driven housing bubbles. Even if the income of home buyers is low and a bubbly equilibrium may not exist, if they have access to sufficient credit, a housing bubble may emerge. During a credit-driven housing bubble, because agents spend all credit on housing purchase, once in the bubbly regime, additional credit ends up increasing the housing price one-for-one with no real effect on consumption allocation, while additional credit does affect consumption in the fundamental regime. Thus there is a discontinuous effect of credit on consumption allocation between the fundamental and the bubbly regimes. Moreover, using the two-stage phase transition and uniqueness of equilibrium dynamics, we present expectation-driven housing booms containing a bubble and their collapse. In our model, because agents are forward-looking and housing prices reflect information about future economic conditions, whether bubbles arise or not in equilibrium depends on long run expectations about the income ratio of home buyers. As long as agents expect high incomes in the future, housing prices start rising now and contain a bubble, even if the current income of home buyers is low and the economy appears to stay in the fundamental region. During this dynamics driven by optimistic beliefs, the price-income ratio and the price-rent ratio simultaneously rise, and hence the housing price dynamics may appear unsustainable because prices grow faster than incomes. On the other hand, if these optimistic expectations do not materialize, the bubble collapses. We emphasize that this expectation-driven housing bubbles and their collapse occur as the unique equilibrium outcome.

Our third main result is the welfare analysis. It has been widely believed in the literature that the introduction of a productive non-reproducible asset like land eliminates the dynamic inefficiency in overlapping generations models (McCallum, 1987). We theoretically show that this is not necessarily true: inefficient equilibria can still occur even though the housing and rental markets in our model are perfectly competitive and frictionless and housing serves a role as a non-reproducible asset like land. Intuitively, in the fundamental equilibrium in the bubble possibility regime, as the young get richer, the interest rate falls below the economic growth rate. In this situation, housing prices are too low to absorb savings desired by the young. In other words, housing is not serving as a means of savings with enough returns, which generates inefficiencies. The emergence of housing bubbles driven by optimistic expectations increases returns on savings and absorbs enough savings by raising housing prices. Housing serves as a high return savings vehicle and restores efficiency. Therefore policymakers may have a role in guiding expectations and equilibrium selection.

We emphasize that we obtain these results and draw new insights from what could be called the simplest possible model of housing. We thus see our paper as a fundamental theoretical contribution that could be used as a stepping stone for constructing more realistic models aimed for empirical or quantitative analysis.

Related literature

Our paper is related to the literature on the valuation of housing. Unlike quantitative models reviewed in Piazzesi and Schneider (2016), our primary interest is to study conditions under which housing could be or must be overvalued, and is closer to the literature on monetary models initiated by the seminal papers of Samuelson (1958), Bewley (1980), Tirole (1985), and Scheinkman and Weiss (1986); see Hirano and Toda (2024a) for a recent review of this literature.

Money is often called a "pure bubble" because it generates no dividends and hence is intrinsically worthless. The so-called "rational bubble literature" has almost exclusively focused on pure bubbles due to the fundamental difficulty of attaching bubbles to dividend-paying assets (Santos and Woodford, 1997). Several papers such as Kocherlakota (2009, 2013), Arce and López-Salido (2011), Zhao (2015), and Chen and Wen (2017) examine housing bubbles in this framework. However, in these papers, either housing does not generate housing services or the rental market is missing and housing do not generate rents, so the fundamental value of housing is zero, which is essentially the same as pure bubbles. As Hirano and Toda (2024a, §4.7) argue, in describing bubbles attached to real assets, pure bubble models are subject to criticisms such as (i) the lack of realism due to zero dividends, (ii) the lack of robustness due to equilibrium indeterminacy (i.e., the existence of a continuum of pure bubble equilibria), and (iii) the inability to connect to the large empirical literature that uses dividends to test whether asset prices reflect fundamentals (Shiller, 1981; Phillips and Shi, 2020). Our model circumvents all these issues because the bubble is attached to housing, which yields positive rents. Most importantly, economic implications and insights we can draw are fundamentally different between pure bubbles (money) and bubbles attached to real assets: as the present paper shows, housing bubbles necessarily emerge with economic development. See also discussions in §4.1 for another new insight concerning the method of model construction.

The pioneering work of Wilson (1981, §7) provides the first example of bubbles attached to dividend-paying assets. There are substantial differences between his model and ours. First, unlike Wilson (1981), whose main focus is abstract theory, our main focus is to study housing bubbles. Second, dividends are exogenous in Wilson (1981), while in ours housing rents are endogenously determined. This difference is important because when dividends/rents are endogenously determined in general equilibrium, it is not obvious whether housing prices will grow faster than rents, exhibiting a housing bubble. Third, the analysis of Wilson (1981) is limited to giving an example of the nonexistence of fundamental equilibrium using linear utilities. In contrast, we provide a full analysis of the (non)existence of fundamental equilibria and the necessity of housing bubbles, including local determinacy. The forthcoming paper of Hirano and Toda (2024b) proves the Bubble Necessity Theorem in overlapping generations and infinite-horizon Bewley models. Our model builds on their insight but as far as we are aware, our paper is the first to prove the existence and the necessity of rational bubbles attached to housing yielding positive rents in a dynamic general equilibrium model.

With regard to macro-finance models that show bubbles attached to an asset with positive dividends, Barlevy (2014) and Allen, Barlevy, and Gale (2022) study a risk-shifting model in which borrowers know the risks of their investments better than lenders. This asymmetric information encourages the borrowers to gamble on risky assets, allowing asset prices to exceed fundamentals. In our paper, asymmetric information is not relevant to the formation of housing price bubbles and hence we abstract from it.

The monetary theory literature including seminal contributions by Lagos and Wright (2005), Rocheteau and Wright (2005, 2013), and Lagos, Rocheteau, and Wright (2017) studies money or real assets as a medium of exchange using a search-theoretic approach and liquidity premium is interpreted as bubbles. Our paper studies rational asset price bubbles attached to real assets in a competitive economy following the standard definition of rational bubbles, which is different from liquidity premium. (See Hirano and Toda (2024b, §5.3) for more discussion.) Although the two approaches are different, they are mutually complementary and provide different insights for the determination of asset prices.

2 Model

2.1 Primitives

Time is discrete and indexed by $t = 0, 1, \dots$. We consider a deterministic overlapping generations (OLG) economy in which agents live for two periods (young and old age) and demand a consumption good and housing service. We employ an OLG model because it allows us to capture life-cycle behaviors regarding housing demand in a simple setting.

Commodities, asset, and endowments There are two perishable commodities (consumption good and housing service) and a durable non-reproducible asset (housing stock) in the economy. The housing service is the right to occupy a house between two periods. Every period, one unit of housing stock inelastically produces one unit of housing service. The time t endowment of the consumption good is $a_t > 0$ for the young and $b_t > 0$ for the old. At $t = 0$, the housing stock (whose aggregate supply is normalized to 1) is owned by the old.

Preferences An agent born at time t lives for two periods and has utility function $U(y_t, z_{t+1}, h_t)$, where $y_t > 0$ is consumption when young, $z_{t+1} > 0$ is consumption when old, and $h_t > 0$ is housing service consumed when transitioning from young to old. As usual, we assume that $U : \mathbb{R}^3_{++} \to \mathbb{R}$ is continuously differentiable, has strictly positive first partial derivatives, is strictly quasi-concave, and satisfies Inada conditions to guarantee interior solutions. The initial old care only about their consumption z_0.

Markets We consider an ideal world in which the ownership and occupancy of housing are separated and traded at competitive frictionless markets: agents trade housing (a financial asset) only to store value (transfer resources across time), whereas they purchase housing service (a commodity) only to derive utility.[5]

Let r_t be the price of housing service (rent) and P_t be the housing price (excluding current rent) quoted in units of time t consumption. Let x_t denote the demand for the housing stock. Then the budget constraints of an agent born at

[5] Therefore nothing prevents agents from purchasing a mansion as an investment while renting a campsite to sleep, or vice versa. Owner-occupants can be thought of agents who rent the houses they own to themselves. However, because in our model agents within a generation are homogeneous, in equilibrium each young agent demands one unit of housing and one unit of housing service, so the agents end up being owner-occupants.

time t are

$$\text{Young:} \qquad y_t + P_t x_t + r_t h_t \le a_t, \tag{2.1a}$$

$$\text{Old:} \qquad z_{t+1} \le b_{t+1} + (P_{t+1} + r_{t+1}) x_t. \tag{2.1b}$$

The budget constraint of the young (2.1a) states that the young spend income on consumption, purchase of housing stock, and rent. The budget constraint of the old (2.1b) states that the old consume the endowment and the income from renting and selling housing.

Equilibrium As usual, an equilibrium is defined by individual optimization and market clearing.

Definition 1. A *rational expectations equilibrium* consists of a sequence of prices $\{(P_t, r_t)\}_{t=0}^\infty$ and allocations $\{(x_t, y_t, z_t, h_t)\}_{t=0}^\infty$ such that for each t, (i) (Individual optimization) The young maximize utility $U(y_t, z_{t+1}, h_t)$ subject to the budget constraints (2.1), (ii) (Commodity market clearing) $y_t + z_t = a_t + b_t$, (iii) (Rental market clearing) $h_t = 1$, (iv) (Housing market clearing) $x_t = 1$.

Note that because the old exit the economy, the young are the natural buyers of housing, which explains the housing market clearing condition $x_t = 1$.

2.2 Equilibrium conditions

We derive equilibrium conditions. Using the rental and housing market clearing conditions $h_t = x_t = 1$ and the budget constraint (2.1), we obtain

$$(y_t, z_t) = (a_t - P_t - r_t, b_t + P_t + r_t) = (a_t - S_t, b_t + S_t), \tag{2.2}$$

where $S_t := P_t + r_t$ is total expenditure on housing. Throughout the paper, we refer to P_t as the *housing price* and S_t as the *housing expenditure*. Let

$$R_t := \frac{P_{t+1} + r_{t+1}}{P_t} = \frac{S_{t+1}}{P_t} \tag{2.3}$$

be the implied gross risk-free rate between time t and $t+1$. Then the two budget constraints in (2.1) can be combined into one as

$$y_t + \frac{z_{t+1}}{R_t} + r_t h_t \le a_t + \frac{b_{t+1}}{R_t}. \tag{2.4}$$

Letting $\lambda_t \geq 0$ be the Lagrange multiplier associated with the combined budget constraint (2.4), we obtain the first-order conditions

$$(U_y, U_z, U_h) = \lambda(1, 1/R_t, r_t), \tag{2.5}$$

where $U_y = \partial U/\partial y$ etc. and the utility function is evaluated at

$$(y_t, z_{t+1}, h_t) = (a_t - S_t, b_{t+1} + S_{t+1}, 1). \tag{2.6}$$

Using (2.5), we obtain $1/R_t = U_z/U_y$ and $r_t = U_h/U_y$. Combining these two equations, the definition of R_t in (2.3), and $S_t = P_t + r_t$, we obtain

$$S_{t+1}U_z = S_tU_y - U_h, \tag{2.7}$$

where the partial derivatives of U are evaluated at (2.6). The following theorem establishes the existence and characterization of equilibrium as a solution to a one-dimensional nonlinear difference equation.

Theorem 1 (Existence and characterization of equilibrium)**.** *A rational expectations equilibrium exists. In any equilibrium, we have* $0 < S_t < a_t$ *and*

$$(y_t, z_t) = (a_t - S_t, b_t + S_t), \tag{2.8a}$$
$$P_t = S_t - (U_h/U_y)(a_t - S_t, b_{t+1} + S_{t+1}, 1), \tag{2.8b}$$
$$r_t = (U_h/U_y)(a_t - S_t, b_{t+1} + S_{t+1}, 1), \tag{2.8c}$$
$$R_t = (U_y/U_z)(a_t - S_t, b_{t+1} + S_{t+1}, 1). \tag{2.8d}$$

Proof. The existence of equilibrium follows from the same argument as the proof of Hirano and Toda (2024b, Theorem 1). The characterizations (2.8) follow from the preceding argument. □

By Theorem 1, an equilibrium is fully characterized by the sequence of housing expenditure $\{S_t\}_{t=0}^\infty$. For this reason, we often refer to $\{S_t\}_{t=0}^\infty$ as an equilibrium without specifying each object in Definition 1.

2.3 Definition and characterization of housing bubbles

Following the standard definition of rational bubbles in the literature (Hirano and Toda, 2024a, §2.1), we define a housing bubble by a situation in which the housing price exceeds its fundamental value defined by the present value of rents.

Let $R_t > 0$ be the equilibrium gross risk-free rate. Let $q_t > 0$ be the Arrow-Debreu price of date-t consumption in units of date-0 consumption, so $q_0 = 1$ and $q_t = 1/\prod_{s=0}^{t-1} R_s$. Since by definition $q_{t+1} = q_t/R_t$ holds, using (2.3) we obtain the no-arbitrage condition

$$q_t P_t = q_{t+1}(P_{t+1} + r_{t+1}). \tag{2.9}$$

Iterating (2.9) forward, for all $T > t$ we obtain

$$q_t P_t = \sum_{s=t+1}^{T} q_s r_s + q_T P_T. \tag{2.10}$$

Since $q_s r_s \geq 0$, we have $\sum_{s=t+1}^{\infty} q_s r_s \leq q_t P_t$, so we may define the *fundamental value* of housing by the present value of rents

$$V_t := \frac{1}{q_t} \sum_{s=t+1}^{\infty} q_s r_s. \tag{2.11}$$

Letting $T \to \infty$ in (2.10), we obtain the limit

$$0 \leq \lim_{T\to\infty} q_T P_T = q_t(P_t - V_t). \tag{2.12}$$

When the limit in (2.12) equals 0, we say that the *transversality condition* (for asset pricing) holds and the asset price P_t equals its fundamental value V_t. When $\lim_{T\to\infty} q_T P_T > 0$, we say that the transversality condition fails and the asset price contains a *bubble*. Note that under rational expectations, we have either $P_t = V_t$ for all t or $P_t > V_t$ for all t. Throughout the rest of the paper, we refer to an equilibrium with (without) a housing bubble a *bubbly (fundamental) equilibrium*.

The economic meaning of $\lim_{T\to\infty} q_T P_T$ is that it captures a purely speculative aspect, that is, agents buy housing now for the purpose of resale in the future, rather than for the purpose of receiving rents. $\lim_{T\to\infty} q_T P_T$ captures its impact on the current housing prices. When the transversality condition holds, the aspect of pure speculation becomes negligible and housing prices are determined only by factors that are backed in equilibrium, namely rents. On the other hand, when the transversality condition is violated, equilibrium housing prices contain a purely speculative aspect.

In general, proving the existence or nonexistence of bubbles is challenging because in the limit (2.12), both the Arrow-Debreu price q_t and the housing price P_t are endogenous. Here we discuss two useful results. Because the context does not matter, we consider a general asset that pays dividend $D_t \geq 0$ and trades at price P_t. The first is the following Bubble Characterization Lemma due to Montrucchio (2004).

Lemma 2.1 (Bubble Characterization, Montrucchio, 2004)**.** *If $P_t > 0$ for all t, the asset price exhibits a bubble if and only if $\sum_{t=1}^{\infty} D_t/P_t < \infty$.*

Proof. See Hirano and Toda (2024b, Lemma 2.1). □

Lemma 2.1 is useful because it does not involve the Arrow-Debreu price q_t and provides a necessary and sufficient condition for the existence of bubbles. The second result is the Bubble Necessity Theorem due to Hirano and Toda (2024b). Because a precise statement is cumbersome, here we only provide an intuitive discussion and refer the readers for details to the original paper. Let R be the counterfactual long run autarky interest rate. Let

$$G_d := \limsup_{t\to\infty} D_t^{1/t} \tag{2.13}$$

be the long run dividend growth rate. Let G be the long run economic growth rate. If the bubble necessity condition

$$R < G_d < G \tag{2.14}$$

holds, then all equilibria are asymptotically bubbly, i.e., there are neither fundamental equilibria nor any bubbly equilibria that are asymptotically bubbleless, and the only possible equilibria are ones in which the asset price is non-negligible in the sense that $\liminf_{t\to\infty} G^{-t}P_t > 0$. Although the proof of the Bubble Necessity Theorem is not obvious, the intuition is clear. If a fundamental equilibrium exists, the asset price must grow at the same rate as dividends, which is G_d. If $G_d < G$, the asset price becomes negligible relative to the size of the economy, and hence the allocation approaches autarky. With an autarky interest rate of $R < G_d$, the present value of dividends (and hence the asset price) becomes infinite, which is impossible. Therefore a fundamental equilibrium cannot exist.

3 Housing prices in the long run

In this section we study the long run behavior of equilibrium housing prices.

3.1 Assumptions

To make qualitative predictions, we put more structure by specializing the utility function and endowments. For the remainder of the paper, the following restrictions are in force.

Assumption 1 (Endowments). *There exist $G > 1$, $a, b > 0$, and $T > 0$ such that the endowments are $(a_t, b_t) = (aG^t, bG^t)$ for $t \geq T$.*

Assumption 1 implies that in the long run, the economy exogenously grows at rate $G > 1$ and the income ratio between the young and old is constant. We assume exogenous growth of endowments and fixed supply of housing as the simplest benchmark to illustrate the key mechanism of housing bubbles.[6] In addition, by assuming a fixed supply, housing serves a role as a productive non-reproducible asset like land, which, as we study in §5, provides new insights on efficiency in OLG economies.

Assumption 2 (Utility). *The utility function takes the form*

$$U(y, z, h) = u(c(y, z)) + v(h), \tag{3.1}$$

where (i) the composite consumption $c(y, z)$ is homogeneous of degree 1 and quasi–concave, (ii) the utility of composite consumption is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some $\gamma > 0$ ($u(c) = \log c$ if $\gamma = 1$), and (iii) the utility of housing service satisfies $v' > 0$.

Assumption 2(i) implies that agents (apart from the initial old) care about consumption (y, z) only through the homothetic composite consumption $c(y, z)$, which (together with Assumption 1) allows us to study asymptotically balanced growth paths. Assumption 2(ii) implies that agents have constant elasticity of substitution $1/\gamma > 0$ between consumption and housing service.[7]

Throughout the main text, we focus on the case $\gamma < 1$ (so the elasticity of substitution between consumption and housing $1/\gamma$ exceeds 1) and defer the analysis of the case $\gamma \geq 1$ to Appendix B. There are three reasons for doing so.

[6] In Figure 5 of Appendix C, we document that economic growth is faster than the growth of housing supply. We can extend our model to include endogenous growth, as studied in Hirano, Jinnai, and Toda (2022), and variable housing supply by introducing the construction of new housing.

[7] To see this, consider an agent who maximizes utility $u(c) + v(h)$ subject to the budget constraint $c + \rho h \leq w$, where $\rho > 0$ is the rent measured in units of composite consumption. Letting λ be the Lagrange multiplier, the first-order conditions are $c^{-\gamma} = \lambda$ and $v'(h) = \lambda\rho$. But in equilibrium we have $h = 1$, so $\rho = v'(1)c^{\gamma}$. Log-differentiating both sides, we obtain $-\frac{\partial \log(h/c)}{\partial \log \rho} = \frac{\partial \log c}{\partial \log \rho} = \frac{1}{\gamma}$, so the elasticity of substitution between consumption and housing is $1/\gamma$.

First, $\gamma < 1$ is the empirically relevant case: (Footnote 4). Second, $\gamma = 1$ is a knife-edge case. Third, as we show in Proposition B.1, the equilibrium with $\gamma > 1$ is pathological and counterfactual: the young asymptotically spend all income on housing (purchase and rent); the price-rent ratio converges to zero; and the gross risk-free rate diverges to infinity. Hence the case $\gamma > 1$ is economically irrelevant.

Since by Assumption 2(i) c is homogeneous of degree 1 and quasi-concave, Theorem 3 of Berge (1963, p. 208) implies that c is actually concave. Because we wish to study smooth interior solutions, we further strengthen the assumption on utility as follows.

Assumption 3 (Composite consumption)**.** *The composite consumption $c : \mathbb{R}^2_{++} \to (0, \infty)$ is homogeneous of degree 1, twice continuously differentiable, and satisfies $c_y > 0$, $c_z > 0$, $c_{yy} < 0$, $c_{zz} < 0$, $c_y(0, z) = \infty$, $c_z(y, 0) = \infty$.*

A typical functional form for c satisfying Assumption 3 is the constant elasticity of substitution (CES) specification

$$c(y, z) = \begin{cases} ((1-\beta)y^{1-\sigma} + \beta z^{1-\sigma})^{\frac{1}{1-\sigma}} & \text{if } 0 < \sigma \neq 1, \\ y^{1-\beta} z^{\beta} & \text{if } \sigma = 1, \end{cases} \tag{3.2}$$

where $1/\sigma$ is the elasticity of intertemporal substitution and $\beta \in (0, 1)$ dictates time preference.

3.2 Definition of long run equilibria

In the subsequent analysis, we first focus on the long run behavior of the economy and then examine transitional dynamics driven by expectations. We present two lemmas that are crucial for the subsequent analysis.

Lemma 3.1 (Backward induction)**.** *Suppose Assumptions 2 and 3 hold. If $\mathcal{S}_T = \{S_t\}_{t=T}^{\infty}$ is an equilibrium starting at $t = T$, there exists a unique equilibrium $\mathcal{S}_0 = \{S_t\}_{t=0}^{\infty}$ starting at $t = 0$ that agrees with $\mathcal{S}_T$ for $t \geq T$.*

Lemma 3.1 shows that we may uniquely extend and equilibrium path backward in time, which allows us to focus on the long run behavior of the economy and guarantees the uniqueness of the transitional dynamics. Since by Assumption 1 the endowments eventually grow at a constant rate G, unless otherwise stated, without loss of generality we assume that endowments are given by $(a_t, b_t) = (aG^t, bG^t)$ for all t.

The following lemma bounds the equilibrium rents.

Lemma 3.2 (Bounds on rents)**.** *Suppose Assumptions 1–3 hold and* $\gamma < 1$*. Then in any equilibrium we have* $0 < \limsup_{t\to\infty} G^{-\gamma t} r_t < \infty$*.*

The intuition for Lemma 3.2 is the following. Since endowments grow at rate G and the elasticity of substitution between consumption and housing service is $1/\gamma$, the marginal rate of substitution (which equals rent) must grow at rate G^γ. In general, we have upper and lower bounds on rents because agents need not consume their endowments and can smooth consumption through savings.

By Lemma 3.2, there exist constants $0 < \underline{r} \le \bar{r} < \infty$ such that $r_t \le \bar{r} G^{\gamma t}$ for all t and $\underline{r} G^{\gamma t} \le r_t$ infinitely often. Taking the $1/t$-th power of both sides and letting $t \to \infty$, the long run rent growth rate (2.13) becomes $\limsup_{t\to\infty} r_t^{1/t} = G^\gamma$. This observation is important for applying the Bubble Necessity Theorem later.

We next define the long run equilibrium. By Assumption 2, the equilibrium dynamics (2.7) becomes

$$S_{t+1} c_z = S_t c_y - m c^\gamma, \tag{3.3}$$

where $m := v'(1)$ is the marginal utility of housing service and c is evaluated at $(y_t, z_{t+1}) = (a_t - S_t, b_{t+1} + S_{t+1})$. To study asymptotically balanced growth paths, let $s_t := S_t/a_t = S_t/(aG^t)$ be the housing expenditure normalized by the income of the young. Since c is homogeneous of degree 1, its partial derivatives c_y, c_z are homogeneous of degree 0. Therefore dividing both sides of (3.3) by aG^t, we obtain

$$G s_{t+1} c_z = s_t c_y - m a^{\gamma-1} G^{(\gamma-1)t} c^\gamma, \tag{3.4}$$

where c, c_y, c_z are evaluated at $(y, z) = (1 - s_t, G(w + s_{t+1}))$ for the old to young income ratio $w := b/a$.

When $\gamma < 1$, the difference equation (3.4) explicitly depends on time t (is non-autonomous), which is inconvenient for analysis. To convert it to an autonomous system, define the auxiliary variable $\xi_t = (\xi_{1t}, \xi_{2t})$ by $\xi_{1t} = s_t = S_t/(aG^t)$ and $\xi_{2t} = a^{\gamma-1} G^{(\gamma-1)t}$. Then the one-dimensional non-autonomous nonlinear difference equation (3.4) reduces to the two-dimensional autonomous nonlinear difference equation $\Phi(\xi_t, \xi_{t+1}) = 0$, where

$$\Phi_1(\xi, \eta) = G\eta_1 c_z - \xi_1 c_y + m c^\gamma \xi_2, \tag{3.5a}$$

$$\Phi_2(\xi, \eta) = \eta_2 - G^{\gamma-1} \xi_2 \tag{3.5b}$$

and c, c_y, c_z are evaluated at $(y, z) = (1 - \xi_1, G(w + \eta_1))$ with $w := b/a$. We can now define a long run equilibrium.

Definition 2. A rational expectations equilibrium $\{S_t\}_{t=0}^{\infty}$ is a *long run equilibrium* if the sequence of auxiliary variables $\{\xi_t\}_{t=0}^{\infty}$ is convergent.

If $\xi_t \to \xi$, since $G > 1$ and $\gamma \in (0, 1)$, we have $\Phi(\xi, \xi) = 0$ if and only if $\xi_2 = 0$ and $\xi_1(Gc_z - c_y) = 0$, where c_y, c_z are evaluated at $(y, z) = (1 - \xi_1, G(w + \xi_1))$. Clearly $\xi_f^* := (0, 0)$ is a steady state of Φ, which we refer to as the *fundamental* steady state. In order for Φ to have a nontrivial ($\xi_1 = s > 0$) steady state, which we refer to as the *bubbly* steady state, it is necessary and sufficient that $Gc_z - c_y = 0$.

3.3 (Non)existence of fundamental equilibria

As a benchmark, we start our analysis with the existence, and possibly nonexistence, of fundamental equilibria. By Lemma 3.2, the rent must asymptotically grow at rate G^γ. Hence if the housing price equals its fundamental value (present value of rents), it must also grow at rate G^γ. But since endowments grow faster at rate $G > G^\gamma$, the expenditure share of housing converges to zero in the long run and the consumption allocation becomes autarkic: $(y_t, z_t) \sim (aG^t, bG^t)$. This argument suggests that in any fundamental equilibrium, the interest rate behaves like

$$R_t = \frac{c_y}{c_z}(y_t, z_{t+1}) \sim \frac{c_y}{c_z}(aG^t, bG^{t+1}) = \frac{c_y}{c_z}(1, Gw), \tag{3.6}$$

where $w := b/a$ is the old to young income ratio and we have used the homogeneity of c (Assumption 2(i)). Obviously, for the fundamental value of housing to be finite, the interest rate cannot fall below the rent growth rate G^γ in the long run. This heuristic argument motivates the following (non)existence result.

Theorem 2 ((Non)existence of fundamental equilibria)**.** *Suppose Assumptions 1–3 hold, $\gamma < 1$, and let $m = v'(1)$ and $w = b/a$. Then the following statements are true.*

(i) There exists a unique $w_f^* > 0$ satisfying

$$\frac{c_y}{c_z}(1, Gw_f^*) = G^\gamma. \tag{3.7}$$

(ii) If $w > w_f^*$, there exists a fundamental long run equilibrium. The equilibrium

objects have the order of magnitude

$$(y_t, z_t) \sim (aG^t, awG^t), \tag{3.8a}$$

$$P_t \sim ma^\gamma \frac{G^\gamma c_z}{c_y - G^\gamma c_z} \frac{c^\gamma}{c_y} G^{\gamma t}, \tag{3.8b}$$

$$r_t \sim ma^\gamma \frac{c^\gamma}{c_y} G^{\gamma t} \tag{3.8c}$$

$$R_t \sim \frac{c_y}{c_z} > G^\gamma, \tag{3.8d}$$

where c, c_y, c_z are evaluated at $(y, z) = (1, Gw)$.

(iii) If $w < w_f^*$, there exist no fundamental equilibria. All equilibria are bubbly with $\liminf_{t\to\infty} G^{-t} P_t > 0$.

Although the conclusion that fundamental equilibria may fail to exist is surprising, its intuition is actually straightforward. As discussed above, in any fundamental equilibrium, the consumption allocation is asymptotically autarkic and the interest rate is pinned down as the marginal rate of intertemporal substitution evaluated at the autarkic allocation. Hence the order of magnitude (3.8) immediately follows from the general analysis in Theorem 1. Because both the housing price and rent grow at rate G^γ, the interest rate (which equals the return on housing by no-arbitrage) must exceed G^γ as in (3.8d). Hence, the transversality condition holds and the housing price just reflects the fundamentals. As the young to old income ratio $1/w = a/b$ rises, the autarkic interest rate falls. But it cannot fall below the rent growth rate G^γ, for otherwise the fundamental value would become infinite, which is impossible in equilibrium. Therefore there cannot be any fundamental equilibria if the young are sufficiently rich. The threshold for the nonexistence of fundamental equilibria is determined by equating the marginal rate of intertemporal substitution to the rent growth rate G^γ, which is precisely the condition (3.7).

It is important to recognize the differences in statements (ii) and (iii). All statement (ii) claims is that there exists a fundamental long run equilibrium satisfying the order of magnitude (3.8). It does not rule out the possibility that there are other equilibria that are potentially cyclic or chaotic. In contrast, statement (iii) is much stronger. Under the condition $w < w_f^*$, it claims that no fundamental equilibria can exist at all, regardless of the asymptotic behavior such as convergent, cyclic, or chaotic.[8] The proof of Theorem 2, especially the nonexistence part

[8] The idea of introducing dividends to rule out fundamental steady states in monetary models,

(iii), is not obvious and builds on the Bubble Necessity Theorem of Hirano and Toda (2024b) in abstract OLG economies.

3.4 Existence of bubbly equilibria

Theorem 2 establishes a necessary and sufficient condition for the existence of a fundamental equilibrium. In particular, if the young are sufficiently rich and $w < w_f^*$, fundamental equilibria do not exist and hence bubbles are inevitable. The following theorem provides a necessary and sufficient condition for the existence of a bubbly long run equilibrium.

Theorem 3 (Existence of bubbly long run equilibrium)**.** *Suppose Assumptions 1–3 hold, $\gamma < 1$, and let $m = v'(1)$ and $w = b/a$. Then the following statements are true.*

(i) There exists a unique $w_b^* > w_f^*$ satisfying

$$\frac{c_y}{c_z}(1, Gw_b^*) = G, \tag{3.9}$$

which depends only on G and c. A bubbly steady state exists if and only if $w < w_b^*$, which is uniquely given by $\xi_b^* = (s^*, 0)$ with $s^* = \frac{w_b^* - w}{w_b^* + 1}$.

(ii) For generic $G > 1$ and $w < w_b^*$, there exists a bubbly long run equilibrium. The equilibrium objects have the order of magnitude

$$(y_t, z_t) \sim (a(1 - s^*)G^t, a(w + s^*)G^t), \tag{3.10a}$$

$$P_t \sim as^* G^t, \tag{3.10b}$$

$$r_t \sim ma^\gamma \frac{c^\gamma}{c_y} G^{\gamma t}, \tag{3.10c}$$

$$R_t \sim G, \tag{3.10d}$$

where c, c_y are evaluated at $(y, z) = (1 - s^*, G(w + s^*))$.

(iii) In the bubbly long run equilibrium, there is a housing bubble and the price-rent ratio P_t/r_t diverges to ∞.

We explain the intuition for the following points: (i) Why does the bubbly equilibrium interest rate R equal the economic growth rate G? (ii) Why do the

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the so called "commodity money refinement", goes back to Scheinkman (1978). However, as Lagos et al. (2017, p. 411) acknowledge, commodity money refinement only rules out equilibria converging to the fundamental steady state. Our statement (iii) is much stronger.

young need to be sufficiently rich for the emergence of bubbles? (iii) Why is the condition $\gamma < 1$ important for the emergence of bubbles? The intuition for (i) is the following. In order for a housing bubble to exist in the long run, housing price must asymptotically grow at the same rate G as the economy as in (3.10b): clearly housing price cannot grow faster than G (otherwise the young cannot afford housing); if it grows at a lower rate than G, housing becomes asymptotically irrelevant. Because housing price grows at rate G but the rent grows at rate $G^\gamma < G$, the interest rate (2.3) must converge to G as in (3.10d). The intuition for (ii) is the following. With bubbles, we know $R = G$. Because the young are saving through the purchase of housing, the lowest possible interest rate in the economy is the autarkic interest rate. Therefore for the emergence of bubbles, the autarkic interest rate must be lower than the economic growth rate, or equivalently the young must be sufficiently rich. The condition (3.9), which equates the marginal rate of intertemporal substitution to the growth rate (long run interest rate), determines the income ratio threshold for which such a situation is possible. The intuition for (iii) is the following. With bubbles, we know $R = G$ and the housing price grows at the same rate. Then the no-arbitrage condition (2.3) forces the rents relative to the prices to be negligible (grow slower), for otherwise the interest rate will exceed the housing price growth rate and there will be no bubbles. Thus for the emergence of bubbles, we need $G > G^\gamma$ and hence $\gamma < 1$.

In this bubbly equilibrium, the housing expenditure S_t and rent r_t asymptotically grow at rates G and $G^\gamma < G$, respectively. On the other hand, since the gross risk-free rate (3.10d) converges to G and the rent grows at rate $G^\gamma < G$, the present value of rents—the fundamental value of housing V_t—is finite and grows at rate G^γ. Then the ratio S_t/V_t grows at rate $G^{1-\gamma} > 1$, so the housing price eventually exceeds the fundamental value. Therefore the transversality condition (2.12) fails and there is a housing bubble. Moreover, from a backward induction argument, we will have housing bubbles at all dates.

In the bubbly equilibrium, the housing price grows faster than the rent and is disconnected from fundamentals in the sense that the housing price is asymptotically independent of the preferences for housing. To see this, note that the threshold w_b^* in (3.9) depends only on the growth rate G and the utility of consumption c. Then the steady state s^* depends only on G, c, and incomes (a, b), and so does the asymptotic housing price (3.10b). In particular, the housing price is asymptotically independent of the marginal utility of housing $m = v'(1)$ as well as the elasticity of substitution $1/\gamma$ between consumption and housing. In contrast, the rent (3.10c) does depend on these parameters.

3.5 Uniqueness of equilibria

Although it is natural to focus on equilibria converging to steady states (i.e., long run equilibria), there may be other equilibria. In general, an equilibrium is called *locally determinate* if there are no other equilibria in a neighborhood of the given equilibrium. If a model does not make determinate predictions, its value as a tool for economic analysis is severely limited. Therefore local determinacy of equilibrium is crucial for applications.

It is well known that equilibria in Arrow-Debreu economies are generically locally determinate (Debreu, 1970) but not necessarily so in OLG models (Gale, 1973; Geanakoplos and Polemarchakis, 1991). In our context, local determinacy means that there are no other equilibria converging to the same steady state. However, we already know the uniqueness of steady states, and we also know that Lemma 3.1 allows us to establish global properties of equilibrium. Thus in our model, local determinacy implies equilibrium uniqueness, which justifies comparative statics and dynamics.

The local determinacy of a dynamic general equilibrium model often depends on the elasticity of intertemporal substitution (EIS) defined by

$$\varepsilon(y,z) = -\left(\frac{\mathrm{d}\log(c_y/c_z)}{\mathrm{d}\log(y/z)}\right)^{-1}; \tag{3.11}$$

see the discussion in Flynn et al. (2023). When c is homogeneous of degree 1, we can show that $\varepsilon = \frac{c_y c_z}{c c_{yz}}$ (Lemma A.1). The following proposition provides a sufficient condition for the uniqueness of equilibria.

Proposition 3.1 (Uniqueness of equilibria)**.** *Suppose Assumptions 1–3 hold and* $\gamma < 1$*. Let* $w = b/a$ *and* w_f^*, w_b^* *be as in* (3.7) *and* (3.9)*. Then the following statements are true.*

(i) If $w > w_f^*$*, there exists a unique fundamental long run equilibrium.*

(ii) If $w < w_b^*$ *and the elasticity of intertemporal substitution* (3.11) *satisfies*

$$\frac{1-w_b^*}{2}\frac{1-w/w_b^*}{1+w} < \varepsilon(y,z) \neq \frac{1-w/w_b^*}{1+w} \tag{3.12}$$

at $(y,z) = (1-s^*, G(w+s^*))$ *with* $s^* = \frac{w_b^*-w}{w_b^*+1}$*, then there exists a unique bubbly long run equilibrium.*

Theorem 2 shows that all fundamental long run equilibria are asymptotically equivalent. Proposition 3.1(i) shows that the fundamental equilibrium is actually unique. The right-hand side of (3.12) is less than 1 because $0 < w < w_b^*$. Therefore the left-hand side of (3.12) is less than 1/2. Proposition 3.1(ii) thus states that the bubbly equilibrium in Theorem 3 is locally determinate as long as the elasticity of intertemporal substitution (EIS) is not too much below 1/2.[9]

The intuition for Proposition 3.1 is as follows. Whether the bubbly equilibrium is locally determinate or not depends on the stability of linearized system around the steady state ξ_b^*. It turns out that one eigenvalue is $\lambda_2 := G^{\gamma-1} < 1$, which is stable. The other eigenvalue λ_1 could be greater than 1 in modulus (unstable) or less (stable), depending on the model parameters. We find that as long as the EIS is not too much below 1/2 (namely the left inequality of (3.12) holds) and is distinct from the special value in the right-hand side of (3.12) (in which case linearization is inapplicable due to a singularity), then $|\lambda_1| > 1$ (unstable). Since the dynamics has one endogenous initial condition (because $\xi_0 = (s_0, a^{\gamma-1})$ and the initial young income a is exogenous), the equilibrium is locally determinate: there exists a unique equilibrium path converging to the steady state if a is large enough. Then the existence and uniqueness of equilibrium with arbitrary a follows from the backward induction argument in Lemma 3.1. The same argument applies to the fundamental equilibrium, although in this case we have $\lambda_1 > 1$ regardless of the EIS.

4 Possibility, necessity, and phase transition

Having established the existence and determinacy of equilibria, in this section we further develop the intuition, discuss credit- and expectation-driven housing bubbles, and present comparative dynamics exercises using a numerical example.

4.1 Two-stage phase transition

Theorems 2 and 3 imply that, as the young (more precisely, home buyers) become richer, the economy experiences *two* phase transitions, as illustrated in Figure 1, which shows how the elasticity of substitution between consumption and housing service $1/\gamma$ and young to old income ratio $1/w = a/b$ affect the equilibrium housing price regimes. (The case $1/\gamma \leq 1$ is treated in Appendix B.)

When the young to old income ratio $1/w = a/b$ is below the bubbly equilibrium threshold $1/w_b^*$, the young do not have sufficient purchasing power to drive up

[9]In general equilibrium theory, it is well known that multiple equilibria are possible if the elasticity is low; see Toda and Walsh (2017) for concrete examples and Toda and Walsh (2024) for a recent review.

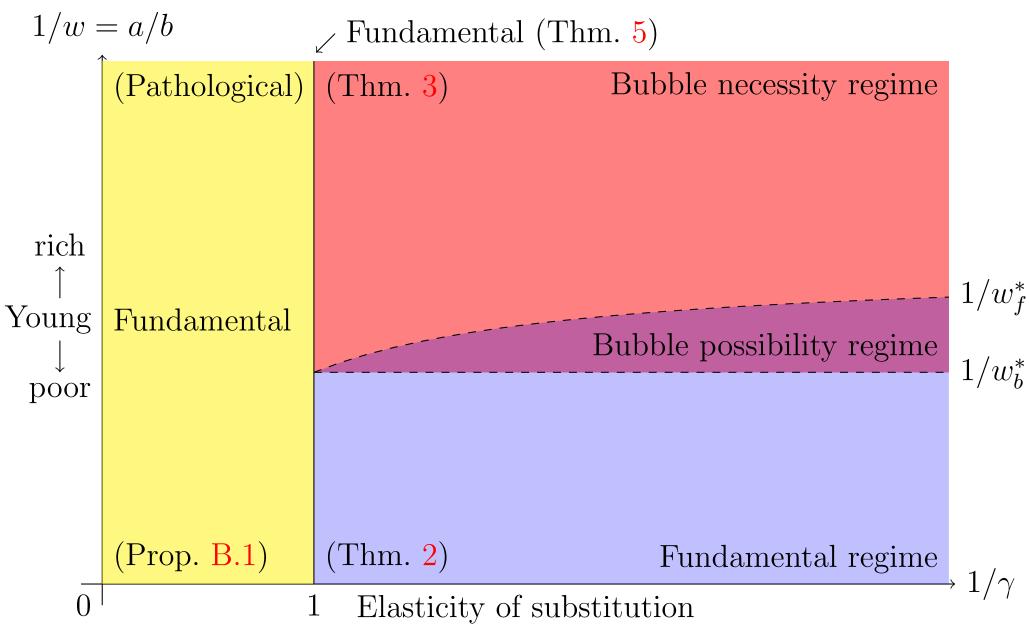


Figure 1: Phase transition of equilibrium housing price regimes.

Note: $1/w = a/b$ is the young to old income ratio and w_f^*, w_b^* are the thresholds for the bubble necessity and possibility regimes defined by (3.7) and (3.9), respectively. The figure corresponds to the CES utility (3.2) with $\beta = 1/2$, $\sigma = 1$, and $G = 1.5$.

the housing price and only fundamental equilibria exist (Theorem 2(ii)). In this fundamental regime, the housing price grows at rate G^γ, which is lower than both the interest rate R and the economic growth rate G. In the long run, the expenditure share of housing converges to zero and the consumption allocation becomes autarkic (see (3.8a)).

When the income ratio of the young exceeds the first critical value $1/w_b^*$, the economy transitions to the bubble possibility regime in which fundamental and bubbly equilibria coexist (Theorem 3). In this regime, although each equilibria are determinate, which equilibrium will be selected depends on agents' expectations.

When the income ratio of the young exceeds the second and still higher critical value $1/w_f^*$, fundamental equilibria cease to exist and all equilibria become bubbly (Theorem 2(iii)). Bubbles are necessary for the existence of equilibrium and the bubble necessity regime emerges. In this regime, the housing price is asymptotically determined only by the economic growth rate G and the preference for consumption goods c, and thus the housing price inevitably becomes disconnected from fundamentals.

The intuition for the necessity of housing bubbles when the young are sufficiently rich is the following. As discussed above, in any fundamental equilibrium, the expenditure share of housing converges to zero and the consumption allocation becomes autarkic. However, as the young get richer (the young to old income ratio $1/w$ increases), the interest rate $R = (c_y/c_z)(1, Gw)$ falls (Figure 2). If R gets lower than a critical value, the economy enters the bubble possibility regime. Hence, housing bubbles driven by optimistic expectations may be possible. As the income ratio increases further, the fundamental equilibrium interest rate becomes lower than the rent growth rate G^γ. If the economy enters that situation, the only possible equilibrium is one that features a housing bubble.

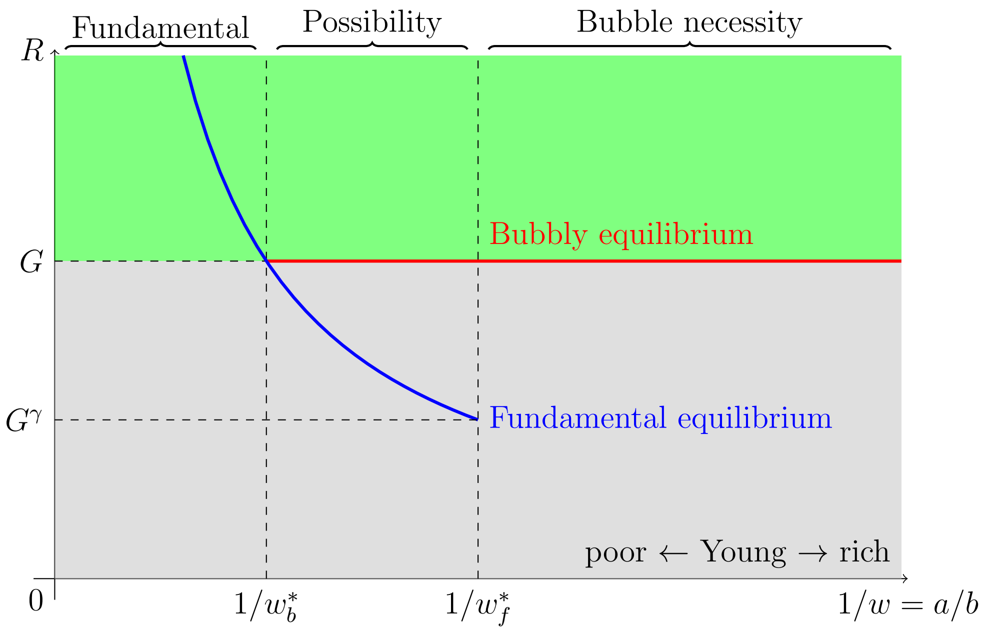


Figure 2: Housing price regimes and equilibrium interest rate.

Note: see Figure 1 for explanation of parameters.

Furthermore, we emphasize that once the state of the economy changes to the housing bubble economy, whether by expectations or by necessity, the determination of housing prices becomes purely demand-driven: the housing price continues to rise due to sustained demand growth arising from income growth of the young (home buyers). In contrast, when housing prices reflect fundamentals, it equals the present value of housing rents and hence its determination is supply-driven. The demand-driven housing price dynamics is a distinctive feature of the housing bubble economy.

We would like to add an important remark concerning the knife-edge case with $\gamma = 1$, i.e., the Cobb-Douglas case, which is often employed in housing models or macroeconomic analyses. When $\gamma = 1$, steady-state growth emerges, in which case housing rents and prices grow at the same rate and therefore housing bubbles are impossible. This result has critically important implications on the method of macroeconomic modelling. As long as we construct a model so that only steady-state growth with stationarity emerges, by model construction, housing bubbles can never occur. As is well known as the Uzawa steady-state growth theorem (Uzawa, 1961; Schlicht, 2006; Jones and Scrimgeour, 2008), obtaining steady-state growth requires knife-edge restrictions. This holds true even in our model. What our analyses show is that once we deviate from the knife-edge restrictions, asset pricing implications become markedly different. This implies that the essence of housing bubbles is nonstationarity. (See also the introduction and concluding remarks in Hirano and Toda (2024a).)

4.2 Credit-driven housing bubbles

So far we have considered a model in which the young self-finance the purchase of housing, but the model can be easily extended to include credit. To see this, let $\left\{(\tilde{a}_t, \tilde{b}_t)\right\}_{t=0}^{\infty}$ be the endowment of some economy with corresponding equilibrium risk-free rate and housing expenditure $\{(R_t, S_t)\}_{t=0}^{\infty}$. Take any sequence $\{\ell_t\}_{t=0}^{\infty}$ such that $\ell_t \in [0, \tilde{a}_t)$ and define $b_0 = \tilde{b}_0$ and $(a_t, b_{t+1}) = (\tilde{a}_t - \ell_t, \tilde{b}_{t+1} + R_t \ell_t)$ for $t \geq 0$. Then we can construct an equilibrium in which the endowment is (a_t, b_t), the interest rate is R_t, the housing expenditure is S_t, and an external banking sector provides loan ℓ_t to the young at time t. We can see this as follows. At time t, the available funds of the young is $a_t + \ell_t = \tilde{a}_t$. At time $t+1$, because the old repay $R_t \ell_t$, the available funds is $b_{t+1} - R_t \ell_t = \tilde{b}_{t+1}$. Therefore given the available funds and the interest rate R_t, it is optimal for the young to spend S_t on housing, so we have an equilibrium.

This argument shows that, even if the income share of the young a_t/b_t is low and a bubbly equilibrium may not exist, if the young have access to sufficient credit, a housing bubble may emerge. In particular, we have the following proposition.

Proposition 4.1. *Let everything be as in Theorem 3 and suppose the banking sector is willing to lend $\ell_t = \ell G^t$ to the young. If the loan to income ratio satisfies*

$$w > \lambda := \frac{\ell}{a} > \frac{w - w_b^*}{w_b^* + 1}, \tag{4.1}$$

then there exists a bubbly long run equilibrium. Under this condition, the housing price has order of magnitude

$$P_t \sim a \left(\frac{w_b^* - w}{w_b^* + 1} + \lambda \right) G^t = a s^* G^t + \ell_t, \tag{4.2}$$

so credit increases the housing price one-for-one.

Using (4.2) and $G^\gamma < G$, by a similar calculation as in (3.10a), the consumption of the young has the order of magnitude

$$\begin{aligned} y_t &= a_t + \ell_t - P_t - r_t \\ &\sim aG^t + \ell_t - (as^* G^t + \ell_t) = a(1-s^*)G^t, \end{aligned}$$

which is independent of credit ℓ_t. Therefore, once the home buyers have access to sufficient credit such that a housing bubble emerges, increasing credit further ends up raising the housing price one-for-one with no real effect on the long run consumption allocation. In contrast, as long as the economy stays in the fundamental regime, the increase in credit does affect the consumption allocation. Thus there is a discontinuous effect of a credit increase on the consumption allocation between two regimes.

4.3 Expectation-driven housing bubbles

We illustrate the preceding analysis and the role of expectations with a numerical example. Suppose the composite consumption takes the CES form (3.2). A straightforward calculation yields

$$c_y = (1-\beta)(y/c)^{-\sigma} \quad \text{and} \quad c_z = \beta(z/c)^{-\sigma}. \tag{4.3}$$

Using (3.7), (3.9), and (4.3), we can solve for the critical values for the existence of fundamental and bubbly equilibria as

$$\frac{1-\beta}{\beta}(Gw_f^*)^\sigma = G^\gamma \iff w_f^* = \left(\frac{\beta}{1-\beta}G^{\gamma-\sigma}\right)^{1/\sigma}, \tag{4.4a}$$

$$\frac{1-\beta}{\beta}(Gw_b^*)^\sigma = G \iff w_b^* = \left(\frac{\beta}{1-\beta}G^{1-\sigma}\right)^{1/\sigma}. \tag{4.4b}$$

Substituting (4.3) into (3.3), we obtain

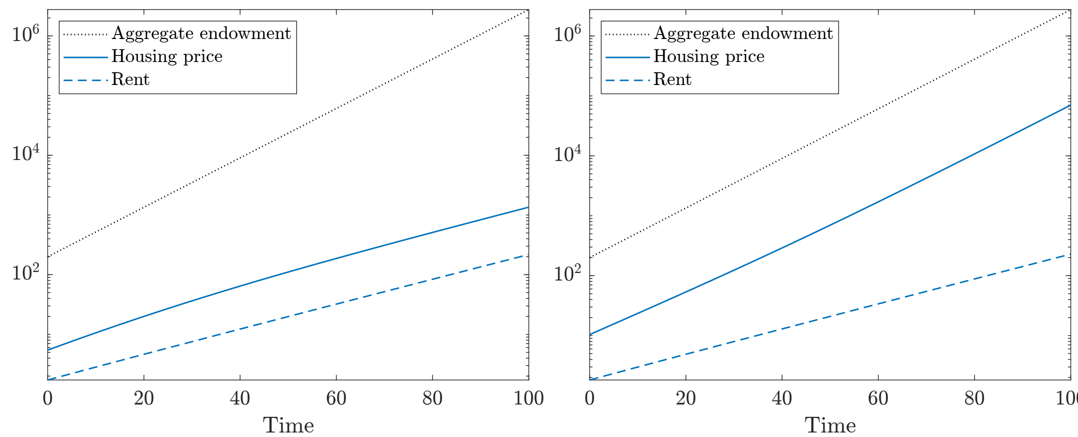
$$\beta S_{t+1} z^{-\sigma} = (1-\beta) S_t y^{-\sigma} - m c^{\gamma-\sigma}, \tag{4.5}$$

where $(y, z) = (a_t - S_t, b_{t+1} + S_{t+1})$. To solve for the equilibrium numerically, we can take a large enough T, set $S_T = s^* a_T$ with steady state value s^* defined by

$$s^* = \begin{cases} 0 & \text{if fundamental equilibrium,} \\ \frac{w_b^* - w}{w_b^* + 1} & \text{if bubbly equilibrium,} \end{cases}$$

and solve the nonlinear equation (4.5) backwards for $S_{T-1}, \dots, S_0$. Note that the backward calculations of $\{S_t\}_{t=0}^T$ are always possible by Lemma 3.1.

As a numerical example, we set $\beta = 1/2$, $\sigma = 1$, $\gamma = 1/2$, $m = 0.1$, and $G = 1.1$. The income ratio threshold for the bubble possibility regime (4.4b) is then $w_b^* = 1$. Figure 3a shows the equilibrium housing price dynamics when $(a, b) = (95, 105)$ so that $b/a > w_b^*$ and hence only a fundamental equilibrium exists. The housing price and rent asymptotically grow at the same rate G^γ, which is lower than the endowment growth rate G. Furthermore, the distance in semilog scale between the housing price and rent converges, suggesting that the price-rent ratio converges. These observations are consistent with Theorem 2.



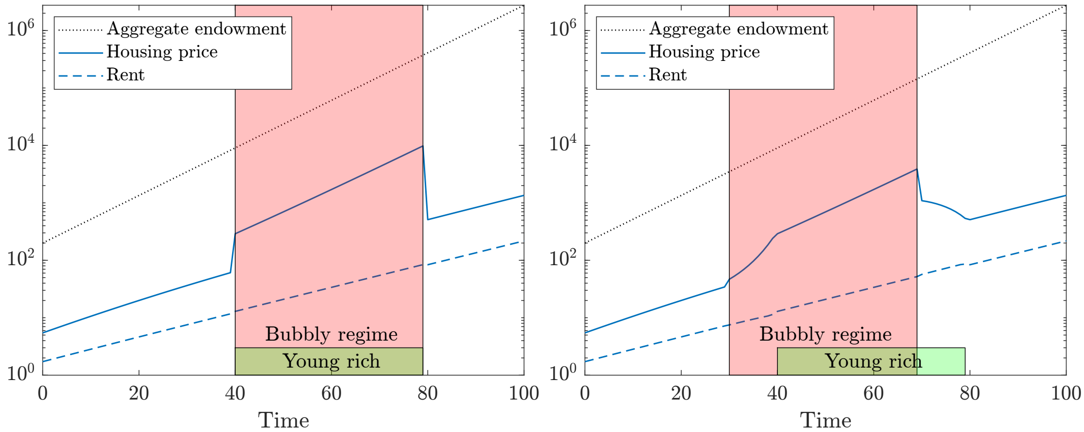
(a) Fundamental equilibrium.

(b) Bubbly equilibrium.

Figure 3: Equilibrium housing price dynamics.

Figure 3b repeats the same exercise for $(a, b) = (105, 95)$ so that $b/a < w_b^*$ and a bubbly equilibrium exists. The housing price asymptotically grows at the same rate as endowments, while the rent grows at a slower rate. Consequently, the price-rent ratio diverges. These observations are consistent with Theorem 3.

We next study how expectations about future incomes affect the current housing price. In Figure 4a, we consider phase transitions between the fundamental and bubbly regimes. The economy starts with $(a_0, b_0) = (95, 105)$ and agents believe that the endowments grow at rate G and the income ratio b_t/a_t is constant at 105/95. At $t = 40$, the income ratio b_t/a_t unexpectedly changes to 95/105 and agents believe that this new ratio will persist. Thus the economy takes off to the bubbly regime. Finally, at $t = 80$ the income ratio b_t/a_t unexpectedly reverts to the original value 105/95. Note that as the economy enters the bubbly regime, rents are hardly affected but the housing price increases and grows at a faster rate, generating a housing bubble.



(a) Unexpected income change. (b) Expected income change.

Figure 4: Phase transition between fundamental and bubbly regimes.

Figure 4b repeats the same exercise except that the income changes are anticipated. Specifically, agents learn at $t = 30$ that the income ratio will change to 95/105 (so the young will be relatively rich) starting at $t = 40$ and will remain so forever. Similarly, agents learn at $t = 70$ that the income ratio will revert to 105/95 (so the young will be relatively poor) starting at $t = 80$ and will remain so forever. In this case, the economy takes off to the bubbly regime at $t = 30$ and reenters the fundamental regime at $t = 70$ due to rational expectations. We can see that the housing price jumps up at $t = 30$ and grows fast even before the fundamentals change. The housing price already contains a bubble, even if the current income of the young is relatively low and appears to be incapable of generating bubbles. This is due to a backward induction argument: if there is a bubble in the future (so (2.12) holds with strict inequality and the transversality condition fails), there is a bubble in every period. Once the young become relatively rich at $t = 40$, the housing price increases at the same rate as endowments, consistent with Theorem 3. During this phase, Irving Fisher would have been right to proclaim that "prices have reached what looks like a permanently high plateau".[10] The housing bubble

[10] *The New York Times*, October 16, 1929, p. 8. URL: https://www.nytimes.com/1929/10/16/archives/fisher-sees-stocks-permanently-high-yale-economist-tells-purchasing.html.

collapses at $t = 70$ when agents learn that the young will be relatively poor in the future, even though the young remain relatively rich until $t = 80$.

From this analysis, we can draw an interesting implication. During expectation-driven housing bubbles, housing prices grow faster than rents. The price-income ratio continues to rise and hence the dynamics may appear unsustainable. Moreover, the greater the time gap between when news of rising incomes arrives ($t = 30$) and when incomes actually start to rise ($t = 40$), the longer the duration of the seemingly unsustainable dynamics. This expectation-driven housing bubbles and their collapse may capture realistic transitional dynamics. For instance, Miles and Monro (2021) emphasize that the decline in the real interest rate has produced large effects on the evolution of housing prices in the U.K. In our model, the (real) interest rate is endogenously determined and is closely related to the income of home buyers. As their income rises and the interest rate falls below the rent growth rate, a housing bubble necessarily emerges. Mankiw and Weil (1989) and Kiyotaki, Michaelides, and Nikolov (2011, 2024) stress the importance of expectation formation of long run aggregate income growth and the interest rate to account for the fluctuations in housing prices. Our expectation-driven housing bubbles and their collapse show that even small changes in incomes of home buyers and/or in their credit availability or the expectation thereof could produce large swings in housing prices. A critical difference is that housing prices in their papers reflect fundamentals, while our main focus is to identify the economic conditions under which housing prices reflect fundamentals or contain bubbles and to study expectation-driven housing price bubbles.[11]

5 Welfare

In §3, we saw that housing bubbles emerge as the young get richer. A natural question is whether housing bubbles are socially desirable or not. It is well known that the competitive equilibrium of an overlapping generations model need not be Pareto efficient (Shell, 1971). This is because OLG models feature double infinity (infinitely many agents and commodities), which could make the present value of aggregate endowments infinite when the interest rate is low and invalidates the usual proof of the First Welfare Theorem. On the other hand, it is known in the literature since McCallum (1987) that the introduction of fiat money or a non-

[11] Beaudry and Portier (2006) show in a macroeconomic model that news about future technological opportunities causes a boom in consumption, investment, hours worked, and stock prices. In their paper, stock prices reflect fundamentals.

reproducible asset such as land resolves the over-savings problem in OLG models and eliminates Pareto inefficient equilibria. We overturn this well-known result: even with the presence of non-reproducible housing, Pareto inefficient equilibria can arise. We will show that it crucially depends on the old to young income ratio.

Let $\{(y_t, z_t, h_t)\}_{t=0}^{\infty}$ be an arbitrary allocation with $y_t, z_t > 0$ and $y_t + z_t = a_t + b_t$. Since only the young demand housing service, which is perishable, it is obviously efficient to assign all housing service to the young. Using Assumption 2, the utility of generation t becomes $U(y_t, z_{t+1}, 1) = u(c(y_t, z_{t+1})) + v(1)$, which is a monotonic transformation of $c(y_t, z_{t+1})$. Therefore the welfare analysis (in terms of Pareto efficiency) reduces to that of an endowment economy without housing and with utility function $c(y, z)$ for goods.

Let $G_t = a_{t+1}/a_t$ be the growth rate of young income and $w_t = b_t/a_t$ be the old to young income ratio at time t. Let $s_t = 1 - y_t/a_t$ be the saving rate. Then the utility of generation t becomes

$$c(y_t, z_{t+1}) = c(a_t(1 - s_t), a_{t+1}(w_{t+1} + s_{t+1})) = a_t c(1 - s_t, G_t(w_{t+1} + s_{t+1})),$$

which is a monotonic transformation of $c(1 - s_t, G_t(w_{t+1} + s_{t+1}))$. This argument shows that the welfare analysis reduces to the case in which the time t aggregate endowment is $1 + w_t$, the utility function of generation t is $u_t(y, z) := c(y, G_t z)$, and the proposed allocation is $(y_t, z_t) = (1 - s_t, w_t + s_t)$. Since Assumption 1 implies that $w_t = b_t/a_t$ is constant for $t \geq T$, we can apply the characterization of Pareto efficiency in OLG models with bounded endowments provided by Balasko and Shell (1980). We thus obtain the following proposition.

Proposition 5.1 (Characterization of equilibrium efficiency). *Suppose Assumptions 1–3 hold and let $\{S_t\}_{t=0}^{\infty}$ be an equilibrium. Let $G_t = a_{t+1}/a_t$, $w_t = b_t/a_t$, and $s_t = S_t/a_t$. Let*

$$R_t = \frac{c_y}{c_z}(1 - s_t, G_t(w_{t+1} + s_{t+1})) \tag{5.1}$$

be the equilibrium risk-free rate and define the Arrow-Debreu price by $q_0 = 1$ and $q_t = 1/\prod_{s=0}^{t-1} R_s$ for $t \geq 1$. Then the following statements are true.

(i) If $\liminf_{t\to\infty} R_t > G$, then the equilibrium is Pareto efficient.

(ii) If $\limsup_{t\to\infty} s_t < 1$, then the equilibrium is Pareto efficient if and only if

$$\sum_{t=0}^{\infty} \frac{1}{G^t q_t} = \infty. \tag{5.2}$$

Proposition 5.1 is an adaptation of Propositions 5.3 and 5.6 of Balasko and Shell (1980) to a growth economy. If we focus on the long run behavior, then $q_t \sim R^{-t}$, so condition (5.2) implies that the equilibrium is efficient if and only if $R \geq G$. We can now apply Proposition 5.1 to determine whether the equilibria in the housing OLG model are efficient or not.

Theorem 4 (Characterization of equilibrium efficiency)**.** *Suppose Assumptions 1–3 hold, $\gamma < 1$, and let $w = b/a$. Then the following statements are true.*

(i) If $w \geq w_b^*$, any equilibrium is efficient.

(ii) If $w < w_b^*$, any bubbly long run equilibrium is efficient.

(iii) If $w < w_b^*$, any fundamental long run equilibrium is inefficient.

Recalling that $w < w_b^*$ implies $R < G$ in the fundamental equilibrium (Figure 2), fundamental equilibria are inefficient whenever $R < G$. Therefore in Figure 2, all equilibria in the green region (including the boundary) are efficient, whereas all equilibria in the gray region (excluding the boundary) are inefficient. This inefficiency result is at odds with the well-known result of McCallum (1987) that the introduction of a productive non-reproducible asset eliminates dynamic inefficiency in OLG models. This is because McCallum (1987) implicitly assumed steady state growth (see his discussion around Endnotes 20 and 21), which does not hold in general. Theorem 4 implies that policymakers may have a role in guiding expectations and equilibrium selection. (See Barlevy (2018) for a discussion of policy issues regarding bubbles.)

The intuition for the Pareto inefficiency of fundamental equilibria when $w < w_b^*$ is the following. In equilibrium, since endowments grow at rate G and the elasticity of substitution between consumption and housing is $1/\gamma$, rents grow at rate G^γ. Therefore if the housing price equals its fundamental value, it must also grow at rate G^γ. Since $G^\gamma < G$, the housing price is asymptotically negligible relative to endowments, so the equilibrium consumption becomes autarkic. Now when $w < w_b^*$, the young are richer, so the interest rate becomes so low that it is below the economic growth rate (see (3.9)). Housing prices are too low to absorb savings desired by the young. In other words, housing is not serving as a means of savings with enough returns. In this situation if we consider a social contrivance such that for each large enough t the young at time t gives the old ϵG^t of the good (hence the old at time $t+1$ receives ϵG^{t+1} of the good), it is as if agents are able to save at rate G higher than the interest rate, which improves welfare. Since this argument holds for all large enough t, we have a Pareto improvement, which implies the inefficiency of the fundamental equilibrium. In addition, when the economy falls into the Pareto inefficient equilibrium with low returns of savings, the emergence of housing bubbles driven by self-fulfilling expectations increases returns on savings and absorbs enough savings by raising housing prices, which restores efficiency. Housing serves as a high return savings vehicle.

6 Concluding remarks

The theory of housing bubbles remains largely underdeveloped due to the fundamental difficulty of attaching bubbles to dividend-paying assets (Santos and Woodford, 1997). In this paper, we have taken the first step towards building a theory of rational housing bubbles. We have presented a bare-bones model of housing bubbles with phase transitions that can be used as a stepping stone for a variety of applications. In concluding our article, we would like to discuss directions for future research.

For analytical tractability and analysis of long-term housing price movements, we based our analysis on the classical overlapping generations model, which is arguably stylized. However, a variety of generalizations are possible, including Bewley-type models with infinitely-lived agents as in Hirano and Toda (2024b, §5). We hope that our bare-bones model of housing bubbles will lead to a variety of extensions both in theoretical and quantitative analyses.

Our theoretical analysis also provides testable implications. First, from the analysis on the long run behavior, housing bubbles are more likely to emerge if the incomes (or available funds through credit) of home buyers are higher or expected to be higher. If the incomes of home buyers rise as economic development progresses, housing bubbles may naturally arise first by optimistic expectations, and then inevitably emerge as the optimistic fundamentals materialize. There is some empirical evidence consistent with this narrative. Gyourko et al. (2013) document that an increase in the high-income population in a metropolitan area is associated with high housing appreciation. Barlevy and Fisher (2021) document that the share of interest-only mortgages is correlated with the housing price growth rates across regions. Second, if there is a housing bubble on the long run trend, rents grow at rate G^γ, whereas housing prices grow at rate G, implying that the price-rent ratio will rise. Hence, an upward trend in the price-rent ratio could be an indicator for housing bubbles. Figure 1 of Amaral et al. (2024) is consistent with this narrative, and the bubble detection literature (Phillips and

Shi, 2020) could be applied. We hope that our theoretical framework may be useful for empirical researchers to investigate these issues further.

A Proofs

A.1 Proof of lemmas

The following lemma lists a few implications of Assumption 3 that will be repeatedly used.

Lemma A.1. *Suppose Assumption 3 holds and let* $g(x) := c(x, 1)$. *Then the following statements are true.*

(i) The first partial derivatives of c *are given by*

$$c_y(y, z) = g'(y/z) > 0, \tag{A.1a}$$
$$c_z(y, z) = g(y/z) - (y/z)g'(y/z) > 0 \tag{A.1b}$$

and are homogeneous of degree 0.

(ii) The second partial derivatives are given by

$$c_{yy}(y, z) = \frac{1}{z}g''(y/z) < 0, \tag{A.2a}$$
$$c_{yz}(y, z) = -\frac{y}{z^2}g''(y/z) > 0, \tag{A.2b}$$
$$c_{zz}(y, z) = \frac{y^2}{z^3}g''(y/z) < 0. \tag{A.2c}$$

(iii) Fixing $z > 0$, *the marginal rate of substitution* c_y/c_z *is continuously differentiable and strictly decreasing in* y *and has range* $(0, \infty)$.

(iv) The elasticity of intertemporal substitution is $\varepsilon(y, z) = \frac{c_y c_z}{c c_{yz}} > 0$.

Proof. By definition, $g(x) = c(x, 1)$. Therefore $g'(x) = c_y(x, 1) > 0$ and $g''(x) = c_{yy}(x, 1) < 0$ by Assumption 3. Since c is homogeneous of degree 1, we have $c(y, z) = zc(y/z, 1) = zg(y/z)$. Then (A.1) and (A.2) are immediate by direct calculation.

Fixing $z > 0$, define the marginal rate of substitution $M(y) = (c_y/c_z)(y, z)$. Then M is continuously differentiable because c is twice continuously differentiable and $c_y, c_z > 0$. Since c_y, c_z are homogeneous of degree 0, we have

$$M(y) = \frac{c_y(y,z)}{c_z(y,z)} = \frac{c_y(y/z,1)}{c_z(1,z/y)}. \tag{A.3}$$

Since $c_y, c_z > 0$ and $c_{yy}, c_{zz} < 0$, the numerator (denominator) is positive and strictly decreasing (increasing) in y. Therefore M is strictly decreasing. Furthermore, since $c_y(0,z) = c_z(y,0) = \infty$, letting $y \downarrow 0$ and $y \uparrow \infty$ in (A.3), we obtain $M(0) = \infty$ and $M(\infty) = 0$, so M has range $(0,\infty)$.

Finally, we derive the elasticity of intertemporal substitution (EIS) ε. Since c is homogeneous of degree 1, we have $c(\lambda y, \lambda z) = \lambda c(y,z)$. Differentiating both sides with respect to λ and setting $\lambda = 1$, we obtain

$$yc_y + zc_z = c. \tag{A.4}$$

Letting $\sigma = 1/\varepsilon$ and $x = y/z$, by the chain rule we obtain

$$\begin{aligned}\sigma &= -\frac{\partial \log(c_y/c_z)(xz,z)}{\partial \log x} = -x\frac{c_z}{c_y}\frac{zc_{yy}c_z - c_y z c_{yz}}{c_z^2}\\ &= y\frac{c_y c_{yz} - c_z c_{yy}}{c_y c_z} = \frac{(yc_y + zc_z)c_{yz}}{c_y c_z} = \frac{cc_{yz}}{c_y c_z},\end{aligned}$$

where the last line uses (A.2) and (A.4). □

Proof of Lemma 3.1. Let $\mathcal{S}_T = \{S_t\}_{t=T}^{\infty}$ be an equilibrium starting at $t = T$. Set $t = T-1$ and define the function $f : [0, a_{T-1}) \to \mathbb{R}$ by $f(S) = S_T c_z - Sc_y + mc^\gamma$, where c, c_y, c_z are evaluated at $(y,z) = (a_{T-1} - S, b_T + S_T)$. Then

$$f'(S) = -S_T c_{yz} - c_y + Sc_{yy} - m\gamma c^{\gamma-1}c_y < 0$$

by Lemma A.1. Clearly $f(0) = S_T c_z + mc^\gamma > 0$. Define

$$\tilde{u}(y,z) := u(c(y,z)) = \begin{cases} \frac{1}{1-\gamma}c(y,z)^{1-\gamma} & \text{if } \gamma \neq 1, \\ \log(c(y,z)) & \text{if } \gamma = 1. \end{cases} \tag{A.5}$$

Take any $\bar{y} > 0$ and let $0 < y < \bar{y}$. Using the chain rule and the monotonicity of c, we obtain

$$\tilde{u}_y(y,z) = c(y,z)^{-\gamma}c_y(y,z) > c(\bar{y},z)^{-\gamma}c_y(y,z) \to \infty \tag{A.6}$$

as $y \downarrow 0$ by Assumption 3. Using the definition of f, we obtain $f(S)c^{-\gamma} = S_T\tilde{u}_z -$

$S\tilde{u}_y + m$. Letting $S \uparrow a_{T-1}$ and using (A.6), we obtain $f(S)c^{-\gamma} \to -\infty$. Hence by the intermediate value theorem, there exists a unique $S_{T-1} \in (0, a_{T-1})$ such that $f(S_{T-1}) = 0$. Therefore there exists a unique equilibrium $\mathcal{S}_{T-1} = \{S_t\}_{t=T-1}^{\infty}$ starting at $t = T-1$ that agrees with $\mathcal{S}_T$ for $t \geq T$. The claim follows from backward induction. □

Proof of Lemma 3.2. Take any equilibrium $\{S_t\}_{t=0}^{\infty}$. Using (2.8c) and Assumption 2, the rent is

$$r_t = m\frac{c^\gamma}{c_y}(aG^t - S_t, bG^{t+1} + S_{t+1}). \tag{A.7}$$

Using the trivial bound $0 \leq S_t \leq aG^t$, noting that c is increasing in both arguments and c_y is decreasing (increasing) in y (z) by Lemma A.1, and using the homogeneity of c and c_y, we obtain

$$r_t \leq m\frac{c(aG^t, (a+b)G^{t+1})^\gamma}{c_y(aG^t, bG^{t+1})} = ma^\gamma \frac{c(1, G(1+w))^\gamma}{c_y(1, Gw)} G^{\gamma t} =: \bar{r}G^{\gamma t}.$$

Dividing both sides by $G^{\gamma t}$ and letting $t \to \infty$, we obtain $\limsup_{t\to\infty} G^{-\gamma t} r_t < \infty$.

We next show

$$\liminf_{t\to\infty} G^{-t} S_t < a. \tag{A.8}$$

Suppose to the contrary that $\liminf_{t\to\infty} G^{-t}S_t \geq a$. Using the trivial bound $S_t \leq aG^t$, we obtain $\lim_{t\to\infty} G^{-t}S_t = a$. Take $\epsilon > 0$ such that $G^{-t}S_t > a - \epsilon$ for large enough t. Then

$$\frac{r_t}{P_t} = \frac{r_t}{S_t - r_t} \leq \frac{\bar{r}G^{\gamma t}}{(a-\epsilon)G^t - \bar{r}G^{\gamma t}} \sim \frac{\bar{r}}{a-\epsilon} G^{(\gamma-1)t}$$

as $t \to \infty$, so $\sum_{t=1}^{\infty} r_t/P_t < \infty$ because $\gamma < 1$. By the Bubble Characterization Lemma 2.1, there is a bubble. Using (2.8d), the homogeneity of c, and Assumption 3, the equilibrium interest rate satisfies

$$\begin{aligned} R_t &= \frac{c_y}{c_z}(a_t - S_t, b_{t+1} + S_{t+1}) \\ &= \frac{c_y}{c_z}(a - G^{-t}S_t, G(b + G^{-t-1}S_{t+1})) \to \frac{c_y}{c_z}(0, G(a+b)) = \infty \end{aligned}$$

as $t \to \infty$. Therefore for any $R > G$, we can take $T > 0$ such that $R_t \geq R > G$ for $t \geq T$. Letting $q_t > 0$ be the Arrow-Debreu price, it follows that

$$q_t P_t = \left(q_T \prod_{s=T}^{t-1} \frac{1}{R_s}\right) P_t \leq q_T R^{T-t} aG^t = aq_T R^T (G/R)^t \to 0$$

as $t \to \infty$, so the transversality condition holds and there is no bubble, which is a contradiction.

Finally, let us show $\limsup_{t\to\infty} G^{-\gamma t} r_t > 0$. Since (A.8) holds, we can take $\bar{s} < 1$ such that $S_t/a_t \le \bar{s}$ infinitely often. For such a subsequence, by a similar argument for establishing $\limsup_{t\to\infty} G^{-\gamma t} r_t < \infty$, we obtain

$$r_t \ge m \frac{c((a-a\bar{s})G^t, bG^{t+1})^\gamma}{c_y((a-a\bar{s})G^t, (a+b)G^{t+1})} = ma^\gamma \frac{c(1-\bar{s}, Gw)^\gamma}{c_y(1-\bar{s}, G(1+w))} G^{\gamma t} =: \underline{r} G^{\gamma t}.$$

Dividing both sides by $G^{\gamma t}$ and letting $t \to \infty$, we obtain the claim. □

A.2 Proof of Theorem 2

(i) By Lemma A.1, $(c_y/c_z)(y, G)$ is strictly decreasing in y and has range $(0, \infty)$. Therefore there exists a unique y satisfying $(c_y/c_z)(y, G) = G^\gamma$. Since by Lemma A.1 c_y, c_z are homogeneous of degree 0, we have $(c_y/c_z)(1, G/y) = G^\gamma$, so $w_f^* = 1/y$ uniquely satisfies (3.7).

(ii) We divide the existence proof into several steps.

Step 1. Derivation of an autonomous nonlinear difference equation.

By (3.8b) and (3.8c), if a fundamental long run equilibrium exists, then $S_t = P_t + r_t$ asymptotically grows at rate G^γ. Define the detrended variable $s_t := S_t/(a^\gamma G^{\gamma t})$. Using the homogeneity of c, c_y, c_z, (3.3) implies

$$a^\gamma s_{t+1} G^{\gamma(t+1)} c_z - a^\gamma s_t G^{\gamma t} c_y + m a^\gamma G^{\gamma t} c^\gamma, \tag{A.9}$$

where c, c_y, c_z are evaluated at

$$(y, z) = (1 - s_t a^{\gamma-1} G^{(\gamma-1)t}, G(w + s_{t+1} a^{\gamma-1} G^{(\gamma-1)(t+1)})).$$

Dividing (A.9) by $a^\gamma G^{\gamma t}$ and defining the auxiliary variable $\xi_t = (\xi_{1t}, \xi_{2t}) = (s_t, a^{\gamma-1} G^{(\gamma-1)t})$, it follows that (3.3) can be rewritten as $\Phi(\xi_t, \xi_{t+1}) = 0$, where $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ is given by

$$\Phi_1(\xi, \eta) = G^\gamma \eta_1 c_z - \xi_1 c_y + m c^\gamma, \tag{A.10a}$$
$$\Phi_2(\xi, \eta) = \eta_2 - G^{\gamma-1} \xi_2 \tag{A.10b}$$

and c, c_y, c_z are evaluated at $(y, z) = (1 - \xi_{1t}\xi_{2t}, G(w + \xi_{1,t+1}\xi_{2,t+1}))$.

Step 2. Existence and uniqueness of a fundamental steady state.

If a steady state ξ_f^* of (A.10) exists, it must be $\xi_2 = 0$. Then the steady state condition is

$$G^\gamma s c_z - s c_y + m c^\gamma \iff s = m \frac{c^\gamma}{c_y - G^\gamma c_z},$$

where c, c_y, c_z are evaluated at $(y, z) = (1, Gw)$. For $s > 0$, it is necessary and sufficient that $c_y / c_z > G^\gamma$ at $(y, z) = (1, Gw)$. Since by Lemma A.1 c_y, c_z are homogeneous of degree 0 and c_y/c_z is strictly increasing in z, there exists a fundamental steady state if and only if $w > w_f^*$.

Step 3. Existence and local determinacy of equilibrium.

Define Φ by (A.10) and write $s = s^*$ to simplify notation. Noting that $\xi_f^* = (s^*, 0)$, a straightforward calculation yields

$$D_\xi \Phi(\xi_f^*, \xi_f^*) = \begin{bmatrix} -c_y & -G^\gamma s^2 c_{yz} + s^2 c_{yy} - s m \gamma c^{\gamma-1} c_y \\ 0 & -G^{\gamma-1} \end{bmatrix},$$

$$D_\eta \Phi(\xi_f^*, \xi_f^*) = \begin{bmatrix} G^\gamma c_z & G^{\gamma+1} s^2 c_{zz} - G s^2 c_{yz} + G s m \gamma c^{\gamma-1} c_z \\ 0 & 1 \end{bmatrix},$$

where all functions are evaluated at $(y, z) = (1, Gw)$. Since $D_\eta \Phi$ is invertible, we may apply the implicit function theorem to solve $\Phi(\xi, \eta) = 0$ around $(\xi, \eta) = (\xi_f^*, \xi_f^*)$ as $\eta = \phi(\xi)$, where

$$D\phi(\xi_f^*) = -[D_\eta \Phi]^{-1} D_\xi \Phi = \begin{bmatrix} \frac{c_y}{G^\gamma c_z} & \phi_{12} \\ 0 & G^{\gamma-1} \end{bmatrix}$$

and ϕ_{12} is unimportant. Since $c_y > G^\gamma c_z$, the eigenvalues of $D\phi$ are $\lambda_1 = c_y/(G^\gamma c_z) > 1$ and $\lambda_2 = G^{\gamma-1} \in (0, 1)$. Therefore the steady state ξ_f^* is a saddle point. The Hartman-Grobman theorem (Chicone, 2006, Theorem 4.6), which justifies linearization around the steady state, implies that for any sufficiently large $a > 0$ (so that $\xi_{20} = a^{\gamma-1}$ is close to the steady state value 0), there exists a unique orbit $\{\xi_t\}_{t=0}^\infty$ converging to the steady state ξ_f^*. However, by Assumption 1, choosing a large enough $a > 0$ is equivalent to starting the economy at large enough $t = T$. Lemma 3.1 then implies that there exists a unique equilibrium converging to the steady state regardless of the early endowments $\{(a_t, b_t)\}_{t=0}^{T-1}$.

Step 4. The equilibrium objects have the order of magnitude in (3.8) and the housing price equals its fundamental value.

The order of magnitude (3.8) is obvious from $\lim_{t\to\infty} G^{-t} S_t = 0$, the homogeneity of c, and Theorem 1. In equilibrium, both the housing price P_t and rent r_t

asymptotically grow at rate G^γ. Therefore $\sum_{t=1}^\infty r_t/P_t = \infty$, so there is no bubble by Lemma 2.1.

(iii) We verify the assumptions of the Bubble Necessity Theorem in Hirano and Toda (2024b), henceforth HT. Take any equilibrium, and consider an endowment economy without housing service in which agents have utility $c(y,z)$, the income of the young is $\tilde{a}_t = a_t - r_t$, the income of the old is b_t, and the asset pays dividend r_t. Assumption 4.1 of HT is clearly satisfied. Since $a_t = aG^t$, by Lemma 3.2 we have $\lim_{t\to\infty} \tilde{a}_{t+1}/\tilde{a}_t = G$ and $\lim_{t\to\infty} b_t/\tilde{a}_t = w$, so Assumption 4.2 of HT holds. Since $c(y,z)$ is homogeneous of degree 1, Assumption 4.3 of HT clearly holds. By the remark after Lemma 3.2, the asymptotic rent growth rate is $G_d := G^\gamma$. Finally, since by Lemma A.1 c_y/c_z is strictly decreasing in y (hence strictly increasing in z), if $w < w_f^*$, the autarky interest rate satisfies

$$R = \frac{c_y}{c_z}(1, Gw) < \frac{c_y}{c_z}(1, Gw_f^*) = G^\gamma = G_d < G.$$

Therefore the bubble necessity condition (2.14) is satisfied. By Theorem 2 of HT, there exist no fundamental equilibria, and in fact, all equilibria are bubbly with $\liminf_{t\to\infty} P_t/a_t > 0$. □

A.3 Proof of Theorem 3

We divide the proof into several steps.

Step 1. Existence and uniqueness of a bubbly steady state.

The proof of the existence and uniqueness of w_b^* satisfying (3.9) is identical to Theorem 2(i). Since $G > 1$ and $\gamma < 1$, it follows from (3.7) and (3.9) that

$$(c_y/c_z)(1, Gw_f^*) = G^\gamma < G = (c_y/c_z)(1, Gw_b^*).$$

Since c_y/c_z is strictly increasing in z, we obtain $w_f^* < w_b^*$.

The steady state condition is $Gc_z - c_y = 0$, where c_y, c_z are evaluated at $(y,z) = (1-s, G(w+s))$. Using the homogeneity of c_y, c_z, this condition is equivalent to $(c_y/c_z)(y, G) = G$ for $y = \frac{1-s}{w+s}$, so the bubbly steady state is uniquely determined by

$$\frac{1-s}{w+s} = \frac{1}{w_b^*} \iff s = \frac{w_b^* - w}{w_b^* + 1}. \tag{A.11}$$

Since $s \in (0,1)$, a necessary and sufficient condition for the existence of a bubbly steady state is $w < w_b^*$.

Step 2. Order of magnitude of equilibrium objects and asset pricing implications.

In any equilibrium converging to the bubbly steady state, by definition we have $S_t \sim asG^t$, where $s = s^*$ is the bubbly steady state. Therefore (3.10a) follows from (2.8a). Using (2.8c) and Assumption 2, the rent is

$$r_t = \frac{v'(1)}{u'(c)c_y} = m\frac{c(a_t - S_t, b_{t+1} + s_{t+1})^\gamma}{c_y(a_t - S_t, b_{t+1} + s_{t+1})}. \tag{A.12}$$

Substituting (3.10a) into (A.12) and using the fact that c is homogeneous of degree 1 and c_y is homogeneous of degree 0, we obtain

$$r_t \sim ma^\gamma \frac{c(1-s, G(w+s))^\gamma}{c_y(1-s, G(w+s))} G^{\gamma t},$$

which is (3.10c). Since r_t asymptotically grows at rate $G^\gamma < G$ because $\gamma < 1$, we have $r_t/S_t \to 0$, so $P_t = S_t - r_t \sim S_t$, which is (3.10b). Finally, (3.10d) follows from (2.3), (3.10b), and (3.10c).

Since the housing price P_t and rent r_t asymptotically grow at rates G and $G^\gamma < G$, respectively, the rent-price ratio r_t/P_t decays geometrically at rate $G^{\gamma-1} < 1$. Therefore $\sum_{t=1}^\infty r_t/P_t < \infty$, so there is a housing bubble by Lemma 2.1.

Step 3. Generic existence of equilibrium.

Define Φ by (3.5) and write $s = s^*$ to simplify notation. Noting that $\xi_b^* = (s^*, 0)$, a straightforward calculation yields

$$D_\xi \Phi(\xi_b^*, \xi_b^*) = \begin{bmatrix} -Gsc_{yz} - c_y + sc_{yy} & mc^\gamma \\ 0 & -G^{\gamma-1} \end{bmatrix},$$
$$D_\eta \Phi(\xi_b^*, \xi_b^*) = \begin{bmatrix} Gc_z + G^2sc_{zz} - Gsc_{yz} & 0 \\ 0 & 1 \end{bmatrix},$$

where all functions are evaluated at $(y, z) = (1-s, G(w+s))$. If $D_\eta\Phi$ is invertible, we may apply the implicit function theorem to solve $\Phi(\xi, \eta) = 0$ around $(\xi, \eta) = (\xi_b^*, \xi_b^*)$ as $\eta = \phi(\xi)$, where

$$D\phi(\xi_b^*) = -[D_\eta\Phi]^{-1} D_\xi \Phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & G^{\gamma-1} \end{bmatrix}$$

with

$$\phi_{11} = \frac{Gsc_{yz} + c_y - sc_{yy}}{Gc_z + G^2sc_{zz} - Gsc_{yz}} =: \frac{e}{d} \tag{A.13}$$

and ϕ_{12} is unimportant. Therefore $D\phi(\xi_b^*)$ has two real eigenvalues; one is $\lambda_1 := \phi_{11}$ and the other is $\lambda_2 := G^{\gamma-1} \in (0,1)$ because $G > 1$ and $\gamma \in (0,1)$.

Let us estimate λ_1. Using (A.2), the numerator of (A.13) is

$$\begin{aligned} e &= c_y + s(Gc_{yz} - c_{yy}) = c_y + s\left(-G\frac{y}{z^2}g'' - \frac{1}{z}g''\right) \\ &= c_y - s\frac{Gy+z}{z^2}g'' = c_y - \frac{s(1+w)}{G(w+s)^2}g'', \end{aligned}$$

where we have used $(y,z) = (1-s, G(w+s))$. Similarly, the denominator is

$$\begin{aligned} d &= Gc_z + Gs(Gc_{zz} - c_{yz}) = Gc_z + Gs\left(G\frac{y^2}{z^3}g'' + \frac{y}{z^2}g''\right) \\ &= Gc_z + Gs\frac{y(Gy+z)}{z^3}g'' = Gc_z + \frac{s(1-s)(1+w)}{G(w+s)^3}g''. \end{aligned}$$

At the steady state, we have $Gc_z = c_y = g'$, so

$$e = g' - \frac{s(1+w)}{G(w+s)^2}g'', \qquad d = g' + \frac{s(1-s)(1+w)}{G(w+s)^3}g''. \tag{A.14}$$

Since $s \in (0,1)$ and $g'' < 0$, clearly $e > d$.

We now study each case by the magnitude of d.

Case 1: $d > 0$. If $d > 0$, then $0 < d < e$ and hence $\lambda_1 = e/d > 1$. Since $\lambda_1 > 1 > \lambda_2 > 0$, the steady state ξ_b^* is a saddle point. The existence and uniqueness of an equilibrium path converging to the steady state ξ_b^* follows by the same argument as in the proof of Theorem 2.

Case 2: $d = 0$. If $d = 0$, the implicit function theorem is inapplicable and we cannot study the local dynamics by linearization.

Case 3: $d \in (-e, 0)$. If $-e < d < 0$, then $\lambda_1 = e/d < -1$. Therefore ξ_b^* is a saddle point and there exists a unique equilibrium by the same argument as in the case $d > 0$.

Case 4: $d = -e$. If $d = -e$, then $\lambda_1 = e/d = -1$ and the Hartman-Grobman theorem is inapplicable.

Case 5: $d < -e$. If $d < -e$, then $\lambda_1 = e/d \in (-1, 0)$. Therefore ξ_b^* is a sink and there exist a continuum of equilibria by the same argument as in the case $d > 0$.

In summary, there exists an equilibrium converging to the bubbly steady state except when $d = 0$ or $d = -e$. Therefore for generic G and w, there exists an equilibrium. □

A.4 Proof of propositions

Proof of Proposition 3.1. We have already proved the uniqueness of the fundamental long run equilibrium if $w > w_f^*$ in the proof of Theorem 2.

Suppose $w < w_b^*$. Let $s = \frac{w_b^* - w}{w_b^* + 1}$ be the bubbly steady state and $(y, z) = (1 - s, G(w + s))$. By the proof of Theorem 3, there exists a unique equilibrium converging to the bubbly steady state if $d \in (-e, 0) \cup (0, \infty)$, where d, e are as in (A.14). We rewrite this condition using the EIS defined by $\varepsilon = \frac{c_y c_z}{c c_{yz}}$. Using (A.1), (A.2), (A.4), and $Gc_z = c_y$ at the steady state, we obtain

$$\varepsilon = \frac{c_y c_z}{(yc_y + zc_z)c_{yz}} = \frac{c_y}{(Gy + z)c_{yz}} = -\frac{g'}{g''}\frac{G(w+s)^2}{(1-s)(1+w)}.$$

Therefore (A.14) can be rewritten as

$$e = \left(1 + \frac{1}{\varepsilon}\frac{s}{1-s}\right) g', \qquad d = \left(1 - \frac{1}{\varepsilon}\frac{s}{w+s}\right) g'. \tag{A.15}$$

Since $g' > 0$, we have

$$\begin{aligned} d = 0 &\iff \varepsilon = \frac{s}{w+s} = \frac{1 - w/w_b^*}{1+w}, \\ e + d > 0 &\iff \varepsilon > \frac{s(1 - w - 2s)}{2(1-s)(w+s)} = \frac{1 - w_b^*}{2}\frac{1 - w/w_b^*}{1+w}. \end{aligned}$$

Therefore the sufficient condition (3.12) follows. □

Proof of Proposition 4.1. By the discussion before the proposition, the available funds of the young at time t is $\tilde{a}_t = a_t + \ell_t = (a + \ell)G^t$ and the available funds of the old at time t is $\tilde{b}_t = b_t - G\ell_{t-1} = (b - \ell)G^t$ at interest rate G. Therefore by Theorem 3, a bubbly long run equilibrium exists if

$$0 < \frac{b - \ell}{a + \ell} < w_b^* \iff w > \frac{\ell}{a} > \frac{w - w_b^*}{w_b^* + 1},$$

which is (4.1). Under this condition, because the old to young available funds ratio is $\tilde{w} := \frac{b - \lambda a}{a + \lambda a} = \frac{w - \lambda}{1 + \lambda}$, using (3.10b) we obtain the asymptotic housing price

$$P_t \sim a(1+\lambda)\frac{w_b^* - \tilde{w}}{w_b^* + 1}G^t = a\frac{(1+\lambda)w_b^* - (w - \lambda)}{w_b^* + 1}G^t,$$

which simplifies to (4.2). □

Proof of Proposition 5.1. Let $u_t(y, z) = c(y, G_t z)$ be the utility function in the detrended economy. Then the implied gross risk-free rate at the proposed allocation $(y_t, z_{t+1}) = (1 - s_t, w_{t+1} + s_{t+1})$ is

$$\tilde{R}_t := \frac{u_{ty}}{u_{tz}}(1 - s_t, w_{t+1} + s_{t+1}) = \frac{1}{G_t}\frac{c_y}{c_z}(1 - s_t, w_{t+1} + s_{t+1}) = \frac{R_t}{G_t}.$$

Therefore the Arrow-Debreu price in the detrended economy is $\tilde{q}_t = \prod_{s=0}^{t-1}(G_s/R_s)$.

We now apply the results of Balasko and Shell (1980). If $\liminf_{t\to\infty} R_t > G$, then by Assumption 1 we can take $R > G$ such that $R_t \ge R > G = G_t$ for t large enough. Then $G_t/R_t \le G/R < 1$, so we have $\lim_{t\to\infty} \tilde{q}_t = 0$. Proposition 5.3 of Balasko and Shell (1980) then implies that the equilibrium is efficient.

We next consider the case $\bar{s} := \limsup_{t\to\infty} s_t < 1$. We verify each assumption of Proposition 5.6 of Balasko and Shell (1980). Since the partial derivatives of c can be signed as in Lemma A.1, the Gaussian curvature of indifference curves are strictly positive. Since the time t aggregate endowment of the detrended economy is $1+w_t$, which is bounded by Assumption 1, it follows that the Gaussian curvature of indifference curves within the feasible region (weakly preferred to endowments) is uniformly bounded and bounded away from 0 because $1 - \bar{s} > 0$. Therefore assumptions (a) and (b) hold. Since $\bar{s} < 1$ and G_t, w_{t+1} are bounded, the gross risk-free rate (5.1) can be uniformly bounded from above and away from 0. Therefore assumption (c) holds. Assumption (d) holds because w_t is bounded, and assumption (e) holds because $\liminf_{t\to\infty}(1 - s_t) = 1 - \bar{s} > 0$. Since all assumptions are verified, Proposition 5.6 of Balasko and Shell (1980) implies that the equilibrium is efficient if and only if

$$\infty = \sum_{t=0}^{\infty} \frac{1}{\tilde{q}_t} = \sum_{t=0}^{\infty} \frac{1}{q_t} \prod_{s=0}^{t-1}(1/G_s). \tag{A.16}$$

Since by Assumption 1 we have $G_t = G$ for large enough t, (A.16) is clearly equivalent to (5.2). □

A.5 Proof of Theorem 4

Suppose $\gamma < 1$ and consider any equilibrium. Using (5.1), Assumption 1, Lemma A.1, and $s_t \ge 0$, we obtain

$$R_t = \frac{c_y}{c_z}(1 - s_t, G_t(w_{t+1} + s_{t+1})) \ge \frac{c_y}{c_z}(1, Gw) \tag{A.17}$$

for large enough t. If $w \geq w_b^*$, then (A.17), Lemma A.1, and (3.9) imply

$$R_t \geq \frac{c_y}{c_z}(1, Gw) \geq \frac{c_y}{c_z}(1, Gw_b^*) = G.$$

Since $R_t \geq G$ eventually, the sequence $1/(G^t q_t) = \prod_{s=0}^{t-1}(R_s/G)$ is positive and bounded away from 0. Therefore (5.2) holds, and the equilibrium is efficient.

Suppose $w < w_b^*$ and take any bubbly equilibrium converging to the bubbly steady state. By (3.10b), we can take $p > 0$ such that $P_t \geq pG^t$ for large enough t. Then

$$G^t q_t = \frac{1}{p} q_t p G^t \leq \frac{1}{p} q_t P_t \leq \frac{1}{p} P_0$$

using (2.10). Since $G^t q_t$ is positive and bounded above, $1/(G^t q_t)$ is positive and bounded away from 0, so (5.2) holds and the equilibrium is Pareto efficient.

Suppose $w < w_b^*$ and take the (unique) fundamental equilibrium. Then by Theorem 2 we have $s_t := S_t/(aG^t) \to 0$. Then (5.1), $s_t \to 0$, and $w < w_b^*$ imply that

$$\lim_{t\to\infty} R_t = \frac{c_y}{c_z}(1, Gw) < \frac{c_y}{c_z}(1, Gw_b^*) = G.$$

Therefore we can take $R < G$ and $T > 0$ such that $R_t \leq R < G$ for $t \geq T$. Since

$$\frac{1}{G^t q_t} = \prod_{s=0}^{t-1}(R_s/G) \leq \frac{1}{G^T q_T}(R/G)^{t-T},$$

the sum $\sum_{t=0}^{\infty} 1/(G^t q_t)$ converges to a finite value, so by Proposition 5.1(ii) the equilibrium is inefficient. □

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Online Appendix

B Elasticity of substitution at most 1

The analysis in the main text focused on the empirically relevant case of $\gamma < 1$ (Footnote 4), that is, the elasticity of substitution between consumption and housing $1/\gamma$ exceeds 1. For completeness, we present an analysis for the case $\gamma \geq 1$.

B.1 Elasticity of substitution below 1

We first consider the case $\gamma > 1$, so the elasticity of substitution $1/\gamma$ is less than 1. In this case we cannot study the local dynamics around the steady state by linearization because the implicit function theorem is not applicable due to a singularity. Nevertheless, we may characterize the asymptotic behavior of all equilibria as follows.

Proposition B.1 (Equilibrium with $\gamma > 1$). *Suppose Assumptions 1–3 hold, $\gamma > 1$, and let $w = b/a$. Then the following statements are true.*

(i) In any equilibrium, the equilibrium objects satisfy

$$\lim_{t\to\infty} (y_t, z_t)/(aG^t) = (0, 1 + w), \tag{B.1a}$$

$$\lim_{t\to\infty} P_t/(aG^t) = 0, \tag{B.1b}$$

$$\lim_{t\to\infty} r_t/(aG^t) = 1, \tag{B.1c}$$

$$\lim_{t\to\infty} R_t = \infty. \tag{B.1d}$$

(ii) There is no housing bubble and the price-rent ratio converges to 0.

(iii) Any equilibrium is Pareto efficient.

Proof. Let $\tilde{u}$ be defined by (A.5). Then the equilibrium dynamics (3.4) can be written as

$$Gs_{t+1}\tilde{u}_z = s_t\tilde{u}_y - ma^{\gamma-1}G^{(\gamma-1)t}, \tag{B.2}$$

where $\tilde{u}_y, \tilde{u}_z$ are evaluated at $(y, z) = (1-s_t, G(w+s_{t+1}))$. Define $\underline{s} = \liminf_{t\to\infty} s_t$. Since $s_t \in (0, 1)$, we have $0 \leq \bar{s} \leq 1$. Take a subsequence of (s_t, s_{t+1}) such that

$(s_t, s_{t+1}) \to (\underline{s}, \tilde{s})$ for some $\tilde{s}$. Letting $t \to \infty$ in (B.2) along this subsequence, we obtain

$$0 \le G\tilde{s}\tilde{u}_z(1-\underline{s}, G(w+\tilde{s})) = \underline{s}\tilde{u}_y(1-\underline{s}, G(w+\tilde{s})) - \infty. \tag{B.3}$$

Noting that $\tilde{u}_y(0,z) = \infty$ by (A.6), the only possibility for (B.3) to hold is $\underline{s} = 1$. Then $s_t \to 1$, and

$$\lim_{t\to\infty} \frac{S_t}{aG^t} = \lim_{t\to\infty} s_t = 1. \tag{B.4}$$

Noting that $y_t = aG^t - S_t$ and $z_t = bG^t + S_t$, we obtain (B.1a). Using (2.7) and (2.8c), we obtain

$$r_t = S_t - S_{t+1}\frac{U_z}{U_y} = S_t - S_{t+1}\frac{c_z}{c_y}, \tag{B.5}$$

where c_y, c_z are evaluated at $(y,z) = (1-s_t, G(w+s_{t+1}))$. Dividing both sides of (B.5) by aG^t, letting $t \to \infty$, and using Lemma A.1, we obtain

$$\lim_{t\to\infty} \frac{r_t}{aG^t} = 1 - G \cdot 0 = 1,$$

which is (B.1c). Since $S_t = P_t + r_t$, we immediately obtain (B.1b). Finally, the risk-free rate is

$$R_t = \frac{S_{t+1}}{P_t} = G\frac{S_{t+1}/(aG^{t+1})}{(S_t - r_t)/(aG^t)} \to G\frac{1}{1-1} = \infty,$$

which is (B.1d).

Since $P_t \le S_t \sim aG^t$ grows at rate at most G and the risk-free rate diverges to infinity (hence eventually exceeds the housing price growth rate), the transversality condition holds and there is no housing bubble. Using (B.1b) and (B.1c), we obtain $P_t/r_t \to 0$, so the price-rent ratio converges to 0. The Pareto efficiency of equilibrium follows from (B.1d) and Proposition 5.1(i). □

B.2 Elasticity of substitution equal to 1

We next consider the case $\gamma = 1$ (log utility), which is commonly used in applied theory. When $u(c) = \log c$, the difference equation (3.4) reduces to

$$Gs_{t+1}c_z = s_t c_y - mc, \tag{B.6}$$

which is an autonomous nonlinear implicit difference equation. The following theorem shows that this difference equation admits a unique steady state, which defines a balanced growth path equilibrium.

Theorem 5 (Equilibrium with $\gamma = 1$)**.** *Suppose Assumptions 1–3 hold, $\gamma = 1$, and let $m = v'(1)$ and $w = b/a$. Then the following statements are true.*

(i) There exists a unique steady state $s^* \in (0, 1)$ of (B.6), which depends only on G, w, c, m.

(ii) There exists a unique balanced growth path equilibrium. The equilibrium objects satisfy

$$(y_t, z_t) = (a(1 - s^*)G^t, a(w + s^*)G^t), \tag{B.7a}$$

$$P_t = a\frac{Gs^*c_z}{c_y}G^t, \tag{B.7b}$$

$$r_t = ma\frac{c}{c_y}G^t, \tag{B.7c}$$

$$R_t = \frac{c_y}{c_z} > G, \tag{B.7d}$$

where c, c_y, c_z are evaluated at $(y, z) = (1 - s^*, G(w + s^*))$.

(iii) In the equilibrium (B.7), there is no housing bubble and the price-rent ratio P_t/r_t is constant.

(iv) Any equilibrium converging to the balanced growth path is Pareto efficient.

(v) If in addition the elasticity of intertemporal substitution satisfies

$$\frac{1}{\varepsilon(y,z)} := \frac{cc_{yz}}{c_y c_z} < \frac{1 + w/s^*}{1 + w}\left(1 + Gw\frac{c_z}{c_y}\right) \tag{B.8}$$

at $(y, z) = (1 - s^*, G(w + s^*))$, then the equilibrium is locally determinate.

Proof. We divide the proof into several steps.

Step 1. Existence and uniqueness of s^*.

Letting $s_t = s_{t+1} = s$ in (B.6) and rearranging terms, we obtain the steady state condition

$$Gsc_z = sc_y - mc \iff \frac{Gc_z - c_y}{c} + \frac{m}{s} = 0, \tag{B.9}$$

where c, c_y, c_z are evaluated at $(y, z) = (1 - s, G(w + s))$. Define $f : (0, 1) \to \mathbb{R}$ by

$$f(s) := \log c(1 - s, G(w + s)) + m \log s.$$

Then (B.9) is equivalent to $f'(s) = 0$. Since $s \mapsto (1-s, G(w+s))$ is affine, the logarithmic function is increasing and strictly concave, and $m > 0$, Theorem 4 of Berge (1963, p. 191) implies that f is strictly concave. Clearly $f'(0) = \infty$. Letting $\tilde{u}(y,z) = \log c(y,z)$, an argument similar to the derivation of (A.6) shows $\tilde{u}_y(0,z) = \infty$. Therefore $f'(1) = -\infty$. Since f is strictly concave, it has a unique global maximum $s^* \in (0,1)$, which satisfies $f'(s^*) = 0$ and hence (B.9). Clearly this s^* depends only on G, w, c, m.

Step 2. Existence, uniqueness, and characterization of a balanced growth path.

In any balanced growth path equilibrium, we must have $S_t = as^* G^t$ for some $s^* \in (0,1)$. The previous step establishes the existence and uniqueness of s^*. The consumption allocation (B.7a) follows from (2.8a), and Assumption 1. The rent (B.7c) follows from (2.8c), Assumption 2, and Lemma A.1. Using (B.7c) and (B.9), we obtain the housing price

$$P_t = S_t - r_t = aG^t\left(s - m\frac{c}{c_y}\right) = aG^t\frac{sc_y - mc}{c_y} = aG^t\frac{Gsc_z}{c_y},$$

which is (B.7b). Using (B.9), we obtain the gross risk-free rate

$$R_t = \frac{S_{t+1}}{P_t} = \frac{asG^{t+1}}{aGs(c_z/c_y)G^t} = \frac{c_y}{c_z} = G + \frac{mc}{sc_z} > G,$$

which is (B.7d). Clearly the price-rent ratio is constant by (B.7b) and (B.7c). Since $R > G$, we obtain

$$\lim_{T\to\infty} R^{-T}P_T = \lim_{T\to\infty} a\frac{Gsc_z}{c_y}(G/R)^T = 0,$$

so the transversality condition holds and there is no housing bubble. The Pareto efficiency of equilibrium follows from (B.7d) and Proposition 5.1(i).

Step 3. Sufficient condition for local determinacy of equilibrium.

Define the function $\Phi : (0,1) \times (0,\infty) \to \mathbb{R}$ by

$$\Phi(\xi,\eta) = G\eta c_z - \xi c_y + mc, \tag{B.10}$$

where c, c_y, c_z are evaluated at $(y,z) = (1-\xi, G(w+\eta))$. Then (B.6) can be written as $\Phi(s_t, s_{t+1}) = 0$ and $\Phi(s,s) = 0$ holds, where we write $s = s^*$. Assuming that the implicit function theorem is applicable and partially differentiating (B.10), we can solve the local dynamics as $s_{t+1} = \phi(s_t)$, where

$$\phi'(s) = -\frac{\Phi_\xi}{\Phi_\eta} = -\frac{-Gsc_{yz} + sc_{yy} - (1+m)c_y}{-Gsc_{yz} + G^2 sc_{zz} + G(1+m)c_z}$$
$$= \frac{(1+m)c_y + Gsc_{yz} - sc_{yy}}{G(1+m)c_z - Gsc_{yz} + G^2 sc_{zz}} =: \frac{e}{d}. \tag{B.11}$$

By exactly the same argument as in the proof of Theorem 3, we obtain

$$e = (1+m)c_y - \frac{s(1+w)}{G(w+s)^2} g'',$$
$$d = G(1+m)c_z + \frac{s(1-s)(1+w)}{G(w+s)^3} g''.$$

If $\phi'(s) > 1$, then $s = s^*$ is a source and hence the balanced growth path equilibrium is locally determinate.

We now seek to derive a sufficient condition for local determinacy. Since $g'' < 0$, we have

$$e - d > (1+m)(c_y - Gc_z) = m(1+m)\frac{c}{s} > 0,$$

where we have used (B.9). Therefore if $\Phi_\eta = d > 0$, then $\phi'(s) = e/d > 1$ and we have local determinacy.

Using (B.11), (A.2), and $\sigma := \frac{cc_{yz}}{c_y c_z}$, the sign of Φ_η becomes

$$\begin{aligned}\operatorname{sgn}(\Phi_\eta) &= \operatorname{sgn}\left(-\frac{Gy+z}{z} sc_{yz} + (1+m)c_z\right) \\ &= \operatorname{sgn}\left(-\frac{Gy+z}{z} s\sigma \frac{c_y c_z}{c} + (1+m)c_z\right) \\ &= \operatorname{sgn}\left(-\frac{Gy+z}{z} s\sigma c_y + (1+m)c\right).\end{aligned}$$

Using (A.4) and (B.9), we obtain

$$\operatorname{sgn}(\Phi_\eta) = \operatorname{sgn}\left(-\frac{Gy+z}{z} s\sigma c_y + yc_y + zc_z + sc_y - Gsc_z\right).$$

Substituting $(y, z) = (1-s, G(w+s))$, dividing by $c_y > 0$, and rearranging terms, we obtain

$$\operatorname{sgn}(\Phi_\eta) = \operatorname{sgn}\left(-\frac{G(1+w)}{G(w+s)} s\sigma + 1 + Gw\frac{c_z}{c_y}\right).$$

Therefore we have $\Phi_\eta > 0$ if and only if

$$\frac{1}{\varepsilon} = \sigma < \frac{1 + w/s}{1 + w} \left(1 + Gw\frac{c_z}{c_y}\right),$$

which is exactly (B.8). □

C Stylized facts

This appendix presents stylized facts regarding housing prices and rents.

We use the regional housing price data from `Realtor.com`, which provides detailed monthly data at the county level since July 2016.[12] We use the median listing price in July because the sales volume tends to be higher in spring and summer.

Regional rents are the Fair Market Rents (FMRs) from the U.S. Department of Housing and Urban Development (HUD).[13] FMRs are defined by estimates of 40th percentile gross rents for standard quality units within a metropolitan area or non-metropolitan county and are available for housing units with 0–4 bedrooms. We use the values for three bedrooms.

The number of housing units is "All housing units" in Quarterly Estimates of the Total Housing Inventory for the United States from the Census Bureau,[14] which is available since 1965.

Figure 5 shows the time series of U.S. real GDP and the total number of housing units, where we normalize the values in 1965 to 1. We can see that GDP growth is faster, justifying our assumption $G > 1$ in the model.

Let r_{it} and P_{it} be the rent and housing price in county i in year t constructed above. Figure 6 plots $\log P_{it}$ against $\log r_{it}$ for the year 2023 and estimates

$$\log P_{it} = \alpha + \beta \log r_{it} + \epsilon_{it}, \quad i = 1, \dots, I$$

by ordinary least squares (OLS) regression. The results for other years are all similar. Although this picture only documents correlation, the coefficient $\hat{\beta} = 1.46 > 1$ corresponds to $1/\gamma$ in the model.

[12] https://www.realtor.com/research/data/

[13] https://www.huduser.gov/portal/datasets/fmr.html

[14] https://www.census.gov/housing/hvs/data/histtab8.xlsx

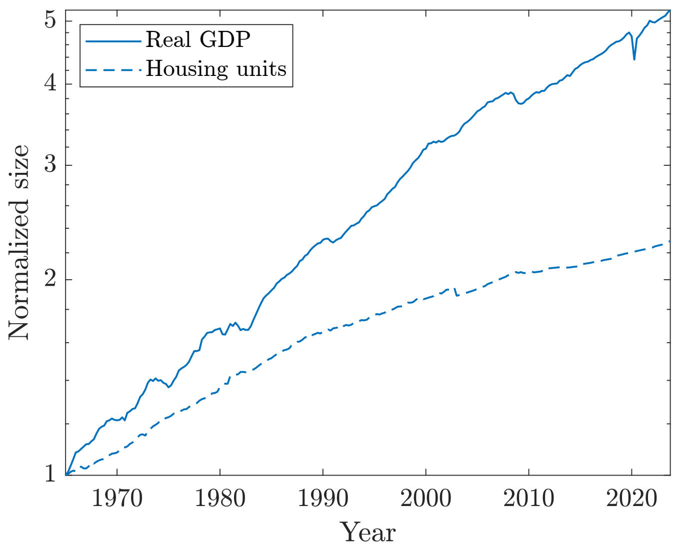

Figure 5: Growth of GDP and housing units.

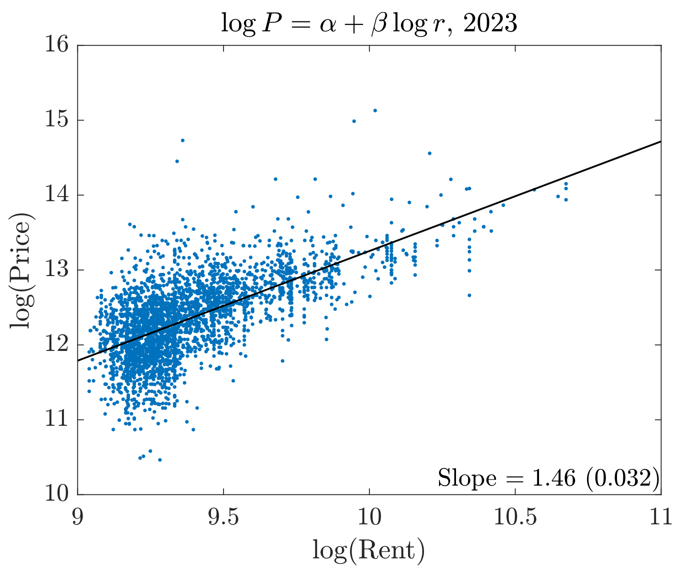

Figure 6: Rent and housing price across counties.