A Risk-centric Model of Demand Recessions and Speculation*

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Abstract

We theoretically analyze the interactions between asset prices, financial speculation, and macroeconomic outcomes when output is determined by aggregate demand. If the interest rate is constrained, a decline in risky asset valuations generates a demand recession. This reduces earnings and generates a negative feedback loop between asset prices and aggregate demand. In the recession phase, beliefs matter not only because they affect asset valuations but also because they determine the strength of the amplification mechanism. In the ex-ante boom phase, belief disagreements (or heterogeneous asset valuations) matter because they induce investors to speculate. This speculation exacerbates the crash by reducing high-valuation investors' wealth when the economy transitions to recession. Macroprudential policy that restricts speculation in the boom can Pareto improve welfare by increasing asset prices and aggregate demand in the recession.

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1. Introduction

Prices of risky assets, such as stocks and houses, fluctuate considerably without meaningful changes in underlying payoffs. These fluctuations, which are due to a host of rational and behavioral mechanisms, are generically described as the result of a "time-varying risk premium" (see, Cochrane (2011); Shiller et al. (2014) and Campbell (2014) for recent reviews). While fluctuations in risky asset prices affect the macroeconomy in a multitude of ways, a growing empirical literature suggests that *aggregate demand* plays a central role and therefore *interest rate policy* can mitigate the impact of asset price shocks (see Pflueger et al. (2018) for evidence that prices of volatile stocks have high predictive power for economic activity and interest rates).[1] A current policy concern is that, with interest rates close to their effective lower bound in much of the developed world, interest rate policy will be unable to respond to future large negative asset price shocks.

This connection between risky asset prices and aggregate demand highlights that *speculation*—a pervasive feature of financial markets driven by heterogeneous asset valuations—can lead to more severe downturns. There is in fact an old tradition in macroeconomics that emphasizes speculation as a central feature of asset prices in boom-bust cycles (see, e.g., Minsky (1977); Kindleberger (1978)). In recent empirical work, Mian and Sufi (2018) argue that speculation also played a key role in the U.S. housing cycle. However, speculation and its interaction with aggregate demand are largely missing from the modern macroeconomic theory connecting asset prices with economic activity, which mostly focuses on financial frictions (see Gertler and Kiyotaki (2010) for a review). This omission is especially important in the current low interest rates environment, as monetary policy has little space to mop up a sharp decline in risky asset prices following a speculative episode.

In this paper, we build a *risk-centric macroeconomic model*—that is, a model in which risky asset prices play an important role—with the two key features highlighted above. First, we emphasize the role of the aggregate demand channel and interest rate frictions in causing recessions driven by a rise in the "risk premium"—our catchall phrase for shocks to asset valuations. Second, we study the impact of financial speculation on the severity of these recessions and derive the implications for macroprudential policy. In order to isolate our insights, we remove all financial frictions.

Our model is set in continuous time with diffusion productivity shocks and Poisson shocks that move the economy between high and low risk premium states. The supply side is a stochastic endowment economy with sticky prices (which we extend to an endogenous growth model when we add investment). The demand side has risk-averse consumer-investors who demand goods and risky assets. We focus on "interest rate frictions" and "financial speculation." By interest rate frictions, we mean factors that might constrain or delay the adjustment of the risk-free interest rate to shocks. For concreteness, we work with a zero lower bound on the policy interest rate, but our mechanism is also applicable with other interest rate constraints such as a currency union or a fixed exchange rate. By financial speculation, we mean the trading of risky financial assets among investors that have heterogeneous valuations for these assets. We capture speculation by allowing

[1] Cieslak and Vissing-Jorgensen (2017) conduct a textual analysis of FOMC minutes and show that the Fed pays attention to stock prices and cuts interest rates after stock price declines ("the Fed put").

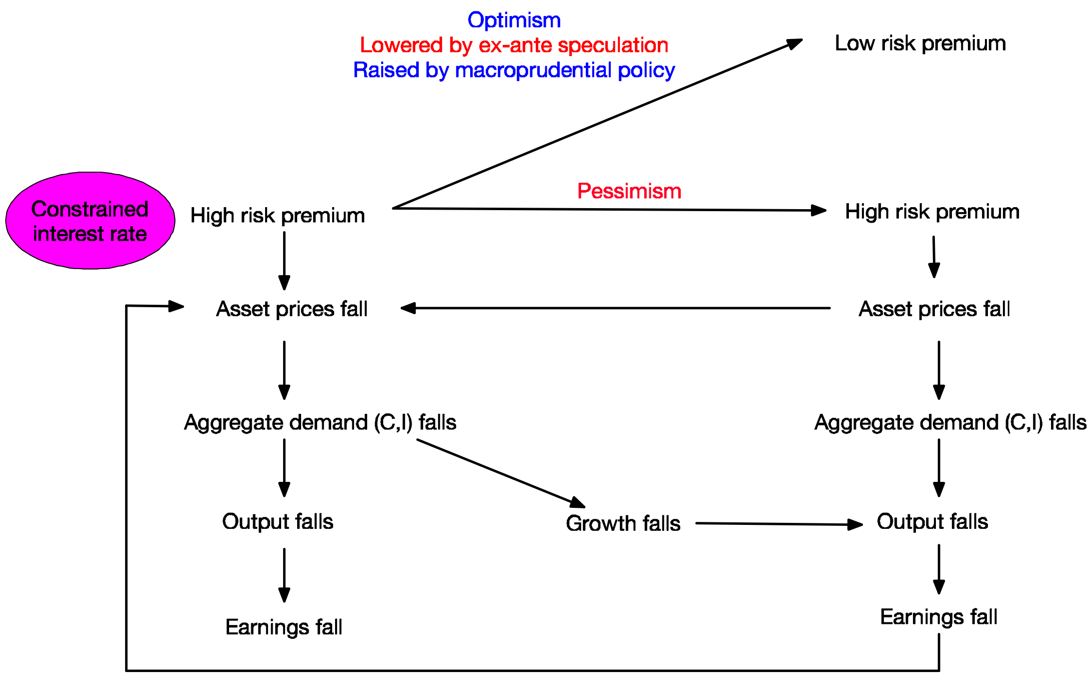

Figure 1: Output-asset price feedbacks during a risk-centric demand recession.

investors to have belief disagreements with respect to the transition probabilities between high and low risk premium states.

To fix ideas, consider an increase in perceived volatility (equivalently, an increase in average pessimism). This is a "risk premium shock" that exerts downward pressure on risky asset prices without a change in current productivity (the supply-determined output level). If the monetary authority allows asset prices to decline, then low prices induce a recession by reducing aggregate demand through a wealth effect. Consequently, monetary policy responds by reducing the interest rate, which stabilizes asset prices and aggregate demand. However, if the interest rate is constrained, the economy loses its natural line of defense. In this case, the rise in the risk premium reduces asset prices and generates a demand recession.

Dynamics play a crucial role in this environment, as the recession is exacerbated by feedback mechanisms. In the main model, when investors expect the higher risk premium to persist, the decline in future demand lowers expected earnings, which exerts further downward pressure on asset prices. With endogenous investment, there is a second mechanism, as the decline in current investment lowers the growth of potential output, which further reduces expected earnings and asset prices. In turn, the decline in asset prices feeds back into current consumption and investment, generating scope for severe spirals in asset prices and output. Figure 1 illustrates these dynamic mechanisms. The feedbacks are especially powerful when investors are *pessimistic* and think the higher risk premium will persist. Hence, beliefs matter in our economy not only because they have a direct impact on asset prices but also because they determine the strength of the amplification mechanism.

In this environment, speculation during the low risk premium phase (boom) exacerbates the recession when there is a transition to the high risk premium phase. With heterogeneous asset valuations, which we capture with belief disagreements about state transition probabilities, the economy's degree of optimism depends on the wealth share of optimists (or high-valuation investors). During recessions, the economy benefits from wealthy optimists because they raise asset valuations, increasing aggregate demand. However, disagreements naturally lead to speculation during booms, which depletes optimists' wealth during recessions. Specifically, optimists take on risk by selling insurance contracts to pessimists that enrich optimists if the boom persists but lead to a large reduction in their wealth share when there is a transition to recession. This reallocation of wealth lowers asset prices and leads to a more severe recession.

These effects motivate *macroprudential policy* that restricts speculation during the boom. We show that macroprudential policy that makes optimistic investors behave as-if they were more pessimistic (implemented via portfolio risk limits) can generate a *Pareto improvement* in social welfare. This result is not driven by paternalistic concerns—the planner respects investors' own beliefs, and the result does not depend on whether optimists or pessimists are closer to the truth. Rather, the planner improves welfare by internalizing *aggregate demand externalities*. The depletion of optimists' wealth during a demand recession depresses asset prices and aggregate demand. Optimists (or more broadly, high-valuation investors) do not internalize the effect of their risk taking on asset prices and aggregate demand during the recession. This leads to excessive risk taking that is corrected by macroprudential policy. Moreover, our model supports *procyclical* macroprudential policy. While macroprudential policy can be useful during the recession, these benefits can be outweighed by its immediate negative impact on asset prices. This decline can be offset by the interest rate policy during the boom but not during the recession.

While there is an extensive empirical literature supporting the components of our model (see Section 7 for a brief summary), we extend this literature by presenting empirical evidence consistent with our results. We focus on three implications. First, our model predicts that shocks to asset valuations generate a more severe demand recession when the interest rate is constrained. Second, the recession reduces firms' earnings and leads to a further decline in asset prices. Third, the recession is more severe when the shock takes place in an environment with more speculation.

To investigate these predictions, we assemble a quarterly panel data set of 21 advanced countries between 1990 and 2017, and subdivide the panel into countries that are part of the Eurozone or the European Exchange Rate Mechanism (*the Euro/ERM sample*) and those that have their own currencies (*the non-Euro/ERM sample*). Countries in the first group have a constrained interest rate with respect to local asset price shocks, since they share a common monetary policy. The second group has a less constrained interest rate. We find that a negative house price shock in a non-Euro/ERM country is associated with an initial decline in economic activity, followed by a decline in the policy interest rate and output stabilization. In contrast, a similar shock in a Euro/ERM country is associated with no interest rate response (compared to other Euro/ERM countries), which is followed by a more persistent and larger decline in economic activity. We also find that the house price shock is followed by a larger decline in earnings and stock prices of publicly traded firms in the Euro/ERM sample (although the standard errors are larger for these results). Finally, we find that past bank credit expansion—which we use as a proxy for speculation on house prices—is associated with more severe outcomes following the house price shock in the Euro/ERM sample (but not in the other sample).

Literature review. Our paper is related to three main literatures: two in macroeconomics and one in finance. On the macroeconomics side, a large body of work emphasizes the links between asset prices and macroeconomic outcomes. Our model contributes to this literature by establishing a relationship between asset prices and aggregate demand even without financial frictions. This relates our paper to strands of the New-Keynesian literature that emphasize demand shocks that might drive business cycles while also affecting asset prices, such as "news shocks" (Beaudry and Portier (2006)), "noise shocks" (Lorenzoni (2009); Blanchard et al. (2013)), "confidence shocks" (Ilut and Schneider (2014)), "uncertainty shocks" (Basu and Bundick (2017); Fernández-Villaverde et al. (2015)), and "disaster shocks" (Isoré and Szczerbowicz (2017)). Aside from the modeling novelty (ours is a continuous time macrofinance model), we provide an integrated treatment of these and related forces. We refer to them as "risk premium shocks" to emphasize their close connection with asset prices and the finance literature on time-varying risk premia. Accordingly, we make asset prices the central object in our analysis, breaking with convention in the New-Keynesian literature without financial frictions.[2] More substantively, we show that heterogeneity in *asset valuation* matters in these environments. This heterogeneity matters because it leads to speculation that exacerbates demand recessions and provides a distinct rationale for macroprudential regulation.

Another important macroeconomics literature focuses on uncertainty and its role in driving macroeconomic fluctuations (e.g., Bloom (2009); Baker et al. (2016); Bloom et al. (2018)). We contribute to this literature by showing how uncertainty affects aggregate activity through asset prices and their impact on aggregate demand. We also illustrate how uncertainty shocks have stronger effects when monetary policy is constrained, consistent with recent empirical evidence (e.g., Plante et al. (2018)). Finally, we show that ex-ante financial speculation amplifies the damage from uncertainty shocks.

On the finance side, a large literature emphasizes investors' beliefs as a key driver of financial boom-bust cycles (see, e.g., Gennaioli and Shleifer (2018) for the role of beliefs in the recent crisis). A strand of this literature argues that heterogeneity in the degree of optimism combined with short-selling constraints can lead to *speculative asset price bubbles* that substantially amplify the financial cycle (e.g., Harrison and Kreps (1978); Scheinkman and Xiong (2003); Geanakoplos (2010); Simsek (2013a); Barberis et al. (2018)). Related contributions emphasize that disagreements exacerbate asset price fluctuations more broadly—even without short-selling constraints or bubbles—because they create endogenous fluctuations in agents' wealth distribution (e.g., Basak (2000, 2005); Detemple and Murthy (1994); Zapatero (1998); Cao (2017); Xiong and Yan (2010);

[2] For an exception, see Galí (2018) who develops an OLG variant of the New-Keynesian model with rational bubbles (see also Biswas et al. (2018)).

Kubler and Schmedders (2012); Korinek and Nowak (2016)). Our paper features similar forces but explores them in an environment where output is not necessarily at its supply-determined level. We show that speculation during the boom not only worsens the asset price bust but also exacerbates the demand recession. Consequently, and unlike much of this literature, macroprudential policy that restricts speculation can improve welfare even if the planner is *not paternalistic* and respects investors' (heterogeneous and possibly over-optimistic) beliefs. Adding paternalistic concerns would reinforce our normative conclusions (see Section 6).[3]

The interactions between heterogeneous valuations, risk-premia, and interest rate lower bounds are central themes of the literature on structural safe asset shortages and safety traps (see, for instance, Caballero and Farhi (2017); Caballero et al. (2017b)). Aside from emphasizing a broader set of factors that can drive the risk premium (in addition to safe asset scarcity), we contribute to this literature by focusing on dynamics. We analyze the connections between boom and recession phases of recurrent business cycles driven by risk premium shocks. We show that speculation between "optimists" and "pessimists" during the boom exacerbates a future risk-centric demand recession, and derive the implications for macroprudential policy. In contrast, Caballero and Farhi (2017) show how "pessimists" can create a demand recession in otherwise normal times and derive the implications for fiscal policy and unconventional monetary policy.[4]

At a methodological level, our paper belongs to the new continuous time macrofinance literature started by the work of Brunnermeier and Sannikov (2014, 2016a) and summarized in Brunnermeier and Sannikov (2016b) (see also Basak and Cuoco (1998); Adrian and Boyarchenko (2012); He and Krishnamurthy (2012, 2013); Di Tella (2017, 2019); Moreira and Savov (2017); Silva (2016)). This literature highlights the full macroeconomic dynamics induced by financial frictions. While the structure of our economy shares many features with theirs, our model has no financial frictions, and the macroeconomic dynamics stem not from the supply side (relative productivity) but from the aggregate demand side.

Our results on macroprudential policy are related to recent work that analyzes the implications of aggregate demand externalities for the optimal regulation of financial markets. For instance, Korinek and Simsek (2016) show that, in the run-up to deleveraging episodes that coincide with a zero-lower-bound on the interest rate, policies targeted at reducing household leverage can improve welfare (see also Farhi and Werning (2017)). In these papers, macroprudential policy works by reallocating wealth across agents and states so that agents with a higher marginal propensity to consume hold relatively more wealth when the economy is depressed due to deficient demand. The mechanism in our paper is different and works through *heterogeneous asset valuations.* The policy operates by transferring wealth to optimists during recessions, not because optimists spend more

[3] More broadly, our paper is part of a large finance literature that investigates the effect of belief disagreements and speculation on financial markets (e.g., Lintner (1969); Miller (1977); Varian (1989); Harris and Raviv (1993); Chen et al. (2002); Fostel and Geanakoplos (2008); Simsek (2013b); Iachan et al. (2015)).

[4] Our paper is also related to an extensive literature on liquidity traps that has exploded since the Great Recession (see, for instance, Tobin (1975); Krugman (1998); Eggertsson and Woodford (2006); Guerrieri and Lorenzoni (2017); Werning (2012); Hall (2011); Christiano et al. (2015); Rognlie et al. (2018); Midrigan et al. (2016); Bacchetta et al. (2016)).

than other investors, but because they raise asset valuations and induce *all investors to spend more* (while also increasing aggregate investment).[5]

The macroprudential literature beyond aggregate demand externalities is mostly motivated by the presence of pecuniary externalities that make the competitive equilibrium constrained inefficient (e.g., Caballero and Krishnamurthy (2003); Lorenzoni (2008); Bianchi and Mendoza (2018); Jeanne and Korinek (2018)). The friction in this literature is market incompleteness or collateral constraints that depend on asset prices (see Davila and Korinek (2016) for a detailed exposition). We show that a decline in asset prices is damaging not only for the reasons emphasized in this literature, but also because it lowers aggregate demand.

The rest of the paper is organized as follows. In Section 2 we present an example that illustrates the main mechanism and motivates the rest of our analysis. Section 3 presents the general environment and defines the equilibrium. Section 4 characterizes the equilibrium in a benchmark setting with common beliefs and homogeneous asset valuations. This section shows how risk premium shocks can lower asset prices and induce a demand recession, and how the recession is exacerbated by feedback loops between asset prices and aggregate demand. Section 5 characterizes the equilibrium with belief disagreements and heterogeneous asset valuations, and illustrates how speculation exacerbates the recession. Section 6 shows the aggregate demand externalities associated with optimists' risk taking and establishes our results on macroprudential policy. Section 7 presents our empirical analysis and summarizes supporting evidence from the related literature. Section 8 concludes. The (online) appendices contain the omitted derivations and proofs as well as the details of our empirical analysis.

2. A stepping-stone example

Here we present a simple (largely static) example that serves as a stepping stone into our main (dynamic) model. We start with a representative agent setup and illustrate the basic aggregate demand mechanism. We then consider belief disagreements and illustrate the role of speculation.

A two-period risk-centric aggregate demand model. Consider an economy with two dates, $t \in \{0, 1\}$, a single consumption good, and a single factor of production—capital. For simplicity, capital is fixed (i.e., there is no depreciation or investment) and it is normalized to one. *Potential* output is equal to capital's productivity, z_t, but the actual output can be below this level due to a shortage of aggregate demand, $y_t \leq z_t$. For simplicity, we assume output is equal to its potential at the last date, $y_1 = z_1$, and focus on the endogenous determination of output at the previous date, $y_0 \leq z_0$. We assume the productivity at date 1 is uncertain and log-normally distributed so that,

$$\log y_1 = \log z_1 \sim N\left(g - \frac{\sigma^2}{2}, \sigma^2\right). \tag{1}$$

[5] See Farhi and Werning (2016) for a synthesis of some of the key mechanisms that justify macroprudential policies in models that exhibit aggregate demand externalities.

We also normalize the initial productivity to one, $z_0 = 1$, so that g captures the (log) expected growth rate of productivity, and σ captures its volatility.

There are two types of assets. There is a "market portfolio" that represents claims to the output at date 1 (which accrue to production firms as earnings), and a risk-free asset in zero net supply. We denote the price of the market portfolio with Q, and its log return with,

$$r^m(z_1) = \log \frac{z_1}{Q}. \tag{2}$$

We denote the log risk-free interest rate with r^f.

For now, the demand side is characterized by a representative investor, who is endowed with the initial output as well as the market portfolio (we introduce disagreements at the end of the section). At date 0, she chooses how much to consume, c_0, and what fraction of her wealth to allocate to the market portfolio, ω^m (with the residual fraction invested in the risk-free asset). When asset markets are in equilibrium, she will allocate all of her wealth to the market portfolio, $\omega^m = 1$, and her portfolio demand will determine the risk premium.

We assume the investor has Epstein-Zin preferences with the discount factor, $e^{-\rho}$, and the relative risk aversion coefficient (RRA), γ. For simplicity, we set the elasticity of intertemporal substitution (EIS) equal to 1. Later in this section, we will show that relaxing this assumption leaves our conclusions qualitatively unchanged. In the dynamic model, we will simplify the analysis further by setting RRA as well as EIS equal to 1 (which leads to time-separable log utility).

The supply side of the economy is described by New-Keynesian firms that have pre-set fixed prices. These firms meet the available demand at these prices as long as they are higher than their marginal cost (see Appendix B.1.2 for details). These features imply that output is determined by the aggregate demand for goods (consumption) up to the capacity constraint,

$$y_0 = c_0 \leq z_0. \tag{3}$$

Since prices are fully sticky, the real interest rate is equal to the nominal interest rate, which is controlled by the monetary authority. We assume that the interest rate policy attempts to replicate the supply-determined output level. However, there is a lower bound constraint on the interest rate, $r^f \geq 0$. Thus, the interest rate policy is described by, $r^f = \max\left(r^{f*}, 0\right)$, where r^{f*} is the natural interest rate that ensures output is at its potential, $y_0 = z_0$.

To characterize the equilibrium, first note that there is a tight relationship between output and asset prices. Specifically, the assumption on the EIS implies that the investor consumes a fraction of her lifetime income,

$$c_0 = \frac{1}{1+e^{-\rho}}\left(y_0 + Q\right). \tag{4}$$

Combining this expression with Eq. (3), we obtain the following equation,

$$y_0 = e^{\rho} Q. \tag{5}$$

We refer to this equation as *the output-asset price relation*—generally, it is obtained by combining the consumption function (and when there is investment, also the investment function) with goods market clearing. The condition says that asset prices increase aggregate wealth and consumption, which in turn leads to greater output.

Next, note that asset prices must also be consistent with equilibrium in risk markets. In Appendix A.1, we show that, up to a local approximation, the investor's optimal weight on the market portfolio is determined by,

$$\omega^m \sigma \simeq \frac{1}{\gamma} \frac{E\left[r^m\left(z_1\right)\right] + \frac{\sigma^2}{2} - r^f}{\sigma}. \tag{6}$$

In words, the optimal portfolio risk (left side) is proportional to "the Sharpe ratio" on the market portfolio (right side). The Sharpe ratio captures the reward per risk, where the reward is determined by the risk premium: the (log) expected return in excess of the (log) risk free rate. This is the standard risk-taking condition for mean-variance portfolio optimization, which applies exactly in continuous time. It applies approximately in the two-period model, and the approximation becomes exact when there is a representative household and the asset markets are in equilibrium ($\omega^m = 1$).

In particular, substituting the asset market clearing condition, $\omega^m = 1$, and the expected return on the market portfolio from Eqs. (1) and (2), we obtain the following equation,

$$\sigma = \frac{1}{\gamma} \frac{g - \log Q - r^f}{\sigma}. \tag{7}$$

We refer to this equation as *the risk balance condition*—generally, it is obtained by combining investors' optimal portfolio allocations with asset market clearing and the equilibrium return on the market portfolio. It says that, the equilibrium level of the Sharpe ratio on the market portfolio (right side) needs to be sufficiently large to convince investors to hold the risk generated by the productive capacity (left side).

Next, consider the supply-determined equilibrium in which output is equal to its potential, $y_0 = z_0 = 1$. Eq. (5) reveals that this requires the asset price to be at a particular level, $Q^* = e^{-\rho}$. Combining this with Eq. (7), the interest rate also needs to be at a particular level,

$$r^{f*} = g + \rho - \gamma\sigma^2. \tag{8}$$

Intuitively, the monetary policy needs to set the interest rate to a low enough level to induce sufficiently high asset prices and aggregate demand to clear the goods market.

Now suppose the initial parameters are such that $r^{f*} > 0$, so that the equilibrium features Q^*, r^{f*} and supply-determined output, $y_0 = z_0 = 1$. Consider a "risk premium shock" that raises the volatility, σ, or risk aversion, γ. The immediate impact of this shock is to create an imbalance in the risk balance condition (7). The economy produces too much risk (left side) relative to what investors are willing to absorb (right side). In response, the monetary policy lowers the risk-free interest rate (as captured by the decline in r^{f*}), which increases the risk premium and equilibrates the risk balance condition (7). Intuitively, the monetary authority lowers the opportunity cost of risky investment and induces investors to absorb risk.

Next suppose the shock is sufficiently large so that the natural interest rate becomes negative, $r^{f*} < 0$, and the actual interest rate becomes constrained, $r^f = 0$. In this case, the risk balance condition is re-established with a decline in the price of the market portfolio, Q. This increases the expected return on risky investment, which induces investors to absorb risk. However, the decline in Q reduces aggregate wealth and induces a demand-driven recession. Formally, we combine Eqs. (5) and (7) to obtain,

$$\log y_0 = \rho + \log Q \qquad \text{where } \log Q = g - \gamma\sigma^2. \tag{9}$$

Note also that, in the constrained region, asset prices and output become sensitive to beliefs about future prospects. For instance, a decrease in the expected growth rate, g (pessimism)—rational or otherwise—decreases asset prices and worsens the recession. In fact, while we analyzed shocks that raise σ or γ, Eqs. (8) and (9) reveal that shocks that lower g lead to the same effects.

More general EIS. Now consider the same model with the difference that we allow the EIS, denoted by ε, to be different than one. Appendix A.2 analyzes this case and shows that the analogue of the output-asset price relation is given by [cf. Eq. (5)],

$$y_0 = e^{\rho\varepsilon} \left(R^{CE}\right)^{1-\varepsilon} Q. \tag{10}$$

Here, R^{CE} denotes the investor's certainty-equivalent portfolio return that we formally define in the appendix. The expression follows from the fact that consumption is not only influenced by a wealth effect, as in the baseline analysis, but also by substitution and income effects. When $\varepsilon > 1$, the substitution effect dominates. All else equal, a decline in the attractiveness of investment opportunities captured by a reduction in R^{CE} tends to reduce savings and increase consumption, which in turn increases output. Conversely, when $\varepsilon < 1$, the income effect dominates and a decline in R^{CE} tends to increase savings and reduce consumption and output.

We also show that the risk balance condition (7) remains unchanged (because the EIS does not affect the investor's portfolio problem). Furthermore, we derive the equilibrium level of the certainty-equivalent return as,

$$\log R^{CE} = g - \log Q - \frac{1}{2}\gamma\sigma^2. \tag{11}$$

As expected, R^{CE} decreases with the volatility, σ, and the risk aversion, γ.

These expressions illustrate that a risk premium shock that increases σ or γ (or lowers g) affects consumption and aggregate demand through two channels. As before, it exerts a downward influence on asset prices, which reduces consumption through a wealth effect. But in this case it also exerts a downward influence on the certainty-equivalent return, which affects consumption further depending on the balance of income and substitution effects. When $\varepsilon > 1$, the second channel works against the wealth effect because investors substitute towards consumption. When $\varepsilon < 1$, the second channel reinforces the wealth effect.

In Appendix A.2, we complete the characterization of equilibrium and show that the net effect on *aggregate demand* is qualitatively the same as in the baseline analysis *regardless of the level of EIS*. In particular, a risk premium shock that increases γ or σ (or lowers g) reduces r^{f*} (see Eq. (A.9)).[6] When the interest rate is constrained, $r^f = 0$, the shock reduces the equilibrium level of output y_0, as well as the asset price, Q (see Eq. (A.10)). When $\varepsilon > 1$, the substitution effect mitigates the magnitude of these declines but it does not overturn them—that is, the wealth effect ultimately dominates. Since the purpose of our model is to obtain qualitative insights, in the dynamic model we assume $\varepsilon = 1$ and isolate the wealth effect.[7]

Belief disagreements and speculation. Let us go back to the baseline case with $\varepsilon = 1$ and illustrate the role of speculation. Suppose that there are two types of investors with heterogeneous beliefs about productivity growth. Specifically, there are optimists and pessimists that believe $\log z_1$ is distributed according to, respectively, $N\left(g^o - \frac{\sigma^2}{2}, \sigma^2\right)$ and $N\left(g^p - \frac{\sigma^2}{2}, \sigma^2\right)$. We assume $g^o > g^p$ so that optimists perceive greater growth. Beliefs are dogmatic, that is, investors know each others' beliefs and they agree to disagree (and it does not matter for our mechanism whether any of them is closer to truth than the other). Optimists are endowed with a fraction α of the market portfolio and of date 0 output (and pessimists are endowed with the remaining fraction). Hence, α denotes the wealth share of optimists. The rest of the model is unchanged.

Following similar steps to those as in the baseline case, we solve for "rstar" as follows (see Appendix A.3),

$$r^{f*} \simeq \alpha g^o + (1-\alpha)\, g^p + \rho - \gamma\sigma^2. \tag{12}$$

When $r^{f*} < 0$, the interest rate is constrained and $r^f = 0$, so we have a demand recession with,

$$\log y_0 = \rho + \log Q, \text{ where } \log Q \simeq \alpha g^o + (1-\alpha)\, g^p - \gamma\sigma^2. \tag{13}$$

Hence, equilibrium prices and output depend on optimists' wealth share, α. During the recession, increasing α improves outcomes because optimists increase asset prices, which increases aggregate wealth and *everyone*'s spending. In our dynamic model, α will be endogenous because investors will (ex-ante) speculate on their different beliefs. Moreover, speculation will reduce α during the recession because optimists think the risk premium shock is unlikely. This will exacerbate the recession and motivate macroprudential policy. Next, we turn to a formal analysis of dynamics.

[6] The effect of this risk premium shock on Q^* is more subtle (see Eq. (A.8)). When $\varepsilon > 1$, Q^* declines, which means that r^{f*} does not need to fully accommodate the risk premium shock. The reason is that the substitution effect supports current consumption and reduces the burden on wealth to support aggregate demand. The opposite happens when $\varepsilon < 1$, where the substitution effect is dominated by the income effect. In this case Q^* needs to rise to support aggregate demand, which is achieved by a larger decline in r^{f*} following the risk premium shock.

[7] We further simplify the dynamic model by setting $\gamma = 1$ (which leads to log utility), because $\gamma \neq 1$ creates additional dynamic hedging motives that are not central for our analysis (see, e.g., Di Tella (2017)).

3. Dynamic environment and equilibrium

In this section we first introduce our general dynamic environment and define the equilibrium. We then describe the optimality conditions and provide a partial characterization of equilibrium. In subsequent sections we will further characterize this equilibrium in various special cases of interest. Throughout, we simplify the analysis by abstracting away from investment. In Appendix D.1, we extend the environment to introduce investment and endogenous growth. We discuss additional results related to investment at the end of Section 4.

Potential output and risk premium shocks. The economy is set in infinite continuous time, $t \in [0, \infty)$, with a single consumption good and a single factor of production, capital. Let $k_{t,s}$ denote the capital stock at time t and in the aggregate state $s \in S$. Suppose that, when fully utilized, $k_{t,s}$ units of capital produce $Ak_{t,s}$ units of the consumption good. Hence, $Ak_{t,s}$ denotes the potential output in this economy. Capital follows the process,

$$\frac{dk_{t,s}}{k_{t,s}} = gdt + \sigma_s dZ_t. \tag{14}$$

Here, g denotes the expected productivity growth, which is an exogenous parameter in the main text (it is endogenized in Appendix D.1 that introduces investment). The term, dZ_t, denotes the standard Brownian motion, which captures "aggregate productivity shocks."[8]

The states, $s \in S$, differ only in terms of the volatility of aggregate productivity, σ_s. For simplicity, there are only two states, $s \in \{1, 2\}$, with $\sigma_1 < \sigma_2$. State $s = 1$ corresponds to a low-volatility state, whereas state $s = 2$ corresponds to a high-volatility state. At each instant, the economy in state s transitions into the other state $s' \neq s$ according to a Poisson process. We use these volatility shocks to capture the time variation in the risk premium due to various unmodeled factors (see Section 2 for an illustration of how risk, risk aversion, or beliefs play a similar role in our analysis).

Transition probabilities and belief disagreements. We let $\lambda_s^i > 0$ denote the perceived Poisson transition probability in state s (into the other state) according to investor $i \in I$. These probabilities capture the degree of investors' (relative) optimism or pessimism. For instance, greater λ_2^i corresponds to greater optimism because it implies the investor expects the current high-risk-premium conditions to end relatively soon. Likewise, smaller λ_1^i corresponds to greater optimism because it implies the investor expects the current low-risk-premium conditions to persist longer. Belief disagreements provide the only exogenous source of heterogeneity in our model. We first analyze the special case with common beliefs (Section 4) and then investigate belief disagreements and

[8] Note that fluctuations in $k_{t,s}$ generate fluctuations in potential output, $Ak_{t,s}$. We introduce Brownian shocks to capital, $k_{t,s}$, as opposed to total factor productivity, A, since this leads to a slightly more tractable analysis when we extend the model to include investment (see Appendix D). In the main text, we could equivalently introduce the shocks to A and conduct the analysis by normalizing all relevant variables with $A_{t,s}$ as opposed to $k_{t,s}$.

speculation (Section 5). When investors disagree, they have dogmatic beliefs (formally, investors know each others' beliefs and they agree to disagree).

Menu of financial assets. There are three types of financial assets. First, there is a market portfolio that represents a claim on all output (which accrues to production firms as earnings as we describe later). We let $Q_{t,s}k_{t,s}$ denote the price of the market portfolio, so $Q_{t,s}$ denotes the price per unit of capital. We let $r^m_{t,s}$ denote the instantaneous expected return on the market portfolio conditional on no transition. Second, there is a risk-free asset in zero net supply. We denote its instantaneous return by $r^f_{t,s}$. Third, in each state s, there is a contingent Arrow-Debreu security that trades at the (endogenous) price $p^{s'}_{t,s}$ and pays 1 unit of the consumption good if the economy transitions into the other state $s' \neq s$. This security is also in zero net supply and it ensures that the financial markets are dynamically complete.

Price and return of the market portfolio. Absent transitions, the price per unit of capital follows an endogenous but deterministic process,[9]

$$\frac{dQ_{t,s}}{Q_{t,s}} = \mu^Q_{t,s}dt \text{ for } s \in \{1,2\}. \tag{15}$$

Combining Eqs. (14) and (15), the price of the market portfolio (conditional on *no* transition) evolves according to,

$$\frac{d\left(Q_{t,s}k_{t,s}\right)}{Q_{t,s}k_{t,s}} = \left(g + \mu^Q_{t,s}\right)dt + \sigma_s dZ_t.$$

This implies that, absent state transitions, the volatility of the market portfolio is given by σ_s, and its expected return is given by,

$$r^m_{t,s} = \frac{y_{t,s}}{Q_{t,s}k_{t,s}} + g + \mu^Q_{t,s}. \tag{16}$$

Here, $y_{t,s}$ denotes the endogenous level of output at time t. The first term captures the "dividend yield" component of return. The second and third terms capture the (expected) capital gain conditional on no transition, which reflects the expected growth of capital as well as of the price per unit of capital.

Eqs. (15 – 16) describe the prices and returns conditional on *no* state transition. If there is a transition at time t from state s into state $s' \neq s$, then the price per unit of capital jumps from $Q_{t,s}$ to a potentially different level, $Q_{t,s'}$. Therefore, investors that hold the market portfolio experience instantaneous capital gains or losses that are reflected in their portfolio problem.

[9] In general, the price follows a diffusion process and this equation also features an endogenous volatility term, $\sigma^Q_{t,s}dZ_t$. In this model, we have $\sigma^Q_{t,s} = 0$ because we work with complete financial markets, constant elasticity preferences, and no disagreements aside from the probability of state transitions. These features ensure that investors allocate identical portfolio weights to the market portfolio (see Eq. (25) later in the section), which ensures that their relative wealth shares are not influenced by dZ_t. The price per capital can be written as a function of investors' wealth shares so it is also not affected by dZ_t.

Consumption and portfolio choice. There is a continuum of investors denoted by $i \in I$, who are identical in all respects except for their beliefs about state transitions, λ_s^i. They continuously make consumption and portfolio allocation decisions. Specifically, at any time t and state s, investor i has some financial wealth denoted by $a_{t,s}^i$. She chooses her consumption rate, $c_{t,s}^i$; the fraction of her wealth to allocate to the market portfolio, $\omega_{t,s}^{m,i}$; and the fraction of her wealth to allocate to the contingent security, $\omega_{t,s}^{s',i}$. The residual fraction, $1 - \omega_{t,s}^{m,i} - \omega_{t,s}^{s',i}$, is invested in the risk-free asset. For analytical tractability, we assume the investor has log utility. The investor then solves a relatively standard portfolio problem that we formally state in Appendix B.1.1.

Equilibrium in asset markets. Asset markets clear when the total wealth held by investors is equal to the value of the market portfolio before and after the portfolio allocation decisions,

$$\int_I a_{t,s}^i di = Q_{t,s} k_{t,s} \text{ and } \int_I \omega_{t,s}^{m,i} a_{t,s}^i di = Q_{t,s} k_{t,s}. \tag{17}$$

Contingent securities are in zero net supply, which implies,

$$\int_I a_{t,s}^i \omega_{t,s}^{s',i} di = 0. \tag{18}$$

The market clearing condition for the risk-free asset (which is also in zero net supply) holds when conditions (17) and (18) are satisfied.

Nominal rigidities and the equilibrium in goods markets. The supply side of our model features nominal rigidities similar to the standard New Keynesian model. We relegate the details to Appendix B.1.2. There is a continuum of monopolistically competitive production firms that own the capital stock and produce intermediate goods (which are then converted into the final good). For simplicity, these production firms have pre-set nominal prices that never change (see Remark 1 below for the case with partial price flexibility). The firms choose their capital utilization rate, $\eta_{t,s} \in [0,1]$, which leads to output, $y_{t,s} = \eta_{t,s} A k_{t,s}$. We assume firms can increase factor utilization for free until $\eta_{t,s} = 1$ and they cannot increase it beyond this level.

As we show in Appendix B.1.2, these features imply that output is determined by aggregate demand for goods up to the capacity constraint. Combining this with market clearing in goods, output is determined by aggregate consumption (up to the capacity constraint),

$$y_{t,s} = \eta_{t,s} A k_{t,s} = \int_I c_{t,s}^i di, \text{ where } \eta_{t,s} \in [0,1]. \tag{19}$$

Moreover, all output accrues to production firms in the form of earnings.[10] Hence, the market portfolio can be thought of as a claim on all production firms.

[10] In this model, firms own the capital so the division of earnings in terms of return to capital and monopoly profits is indeterminate. Since there is no investment, this division is inconsequential. When we introduce investment in Appendix D, we make additional assumptions to determine how earnings are divided between return to capital and monopoly profits.

Interest rate rigidity and monetary policy. Our assumption that production firms do not change their prices implies that the aggregate nominal price level is fixed. The real risk-free interest rate, then, is equal to the nominal risk-free interest rate, which is determined by the interest rate policy of the monetary authority. We assume there is a lower bound on the nominal interest rate, which we set at zero for convenience,

$$r_{t,s}^{f} \geq 0. \tag{20}$$

The zero lower bound is motivated by the presence of cash in circulation (which we leave unmodeled for simplicity).

We assume that the interest rate policy aims to replicate the level of output that would obtain without nominal rigidities subject to the constraint in (20). Without nominal rigidities, capital is fully utilized, $\eta_{t,s} = 1$ (see Appendix B.1.2). Thus, we assume that the interest rate policy follows the rule,

$$r_{t,s}^{f} = \max\left(0, r_{t,s}^{f,*}\right) \text{ for each } t \geq 0 \text{ and } s \in S. \tag{21}$$

Here, $r_{t,s}^{f,*}$ is recursively defined as the (instantaneous) natural interest rate that obtains when $\eta_{t,s} = 1$ and the monetary policy follows the rule in (21) at all future times and states.

Definition 1. *The equilibrium is a collection of processes for allocations, prices, and returns such that capital evolves according to (14), price per unit of capital evolves according to (15), its instantaneous return is given by (16), investors maximize expected utility (cf. Appendix B.1.1), asset markets clear (cf. Eqs. (17) and (18)), production firms maximize earnings (cf. Appendix B.1.2), goods markets clear (cf. Eq. (19)), and the interest rate policy follows the rule in (21).*

Remark 1 (Partial Price Flexibility)**.** *Our assumption of a fixed aggregate nominal price is extreme. However, allowing nominal price flexibility does not necessarily circumvent the bound in (20). In fact, if monetary policy follows an inflation targeting policy regime, then partial price flexibility leads to price deflation during a demand recession. This strengthens the bound in (20) and exacerbates the recession (see Werning (2012); Korinek and Simsek (2016); Caballero and Farhi (2017) for further discussion, and Footnote 14 for a discussion of how partial price flexibility would also strengthen our results with belief disagreements).*

In the rest of this section, we provide a partial characterization of the equilibrium.

Investors' optimality conditions. We derive these optimality conditions in Appendix B.1.1. In view of log utility, the investor's consumption is a constant fraction of her wealth,

$$c_{t,s}^{i} = \rho a_{t,s}^{i}. \tag{22}$$

Moreover, the investor's weight on the market portfolio is determined by,

$$\omega_{t,s}^{m,i} \sigma_s = \frac{1}{\sigma_s} \left(r_{t,s}^{m} - r_{t,s}^{f} + \lambda_s^{i} \frac{1/a_{t,s'}^{i}}{1/a_{t,s}^{i}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \right). \tag{23}$$

That is, she invests in the market portfolio up to the point at which the risk of her portfolio (left side) is equal to the "Sharpe ratio" of the market portfolio (right side). This is similar to the optimality condition in the two period model (cf. Eq. (6)) with the difference that the dynamic model also features state transitions. Our notion of the Sharpe ratio accounts for potential revaluation gains or losses from state transitions (the term, $\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}$) as well as the adjustment of marginal utility in case there is a transition (the term, $\frac{1/a^i_{t,s'}}{1/a^i_{t,s}}$).[11]

Finally, the investor's optimal portfolio allocation to the contingent securities implies,

$$\frac{p^{s'}_{t,s}}{\lambda^i_s} = \frac{1/a^i_{t,s'}}{1/a^i_{t,s}}. \tag{24}$$

The portfolio weight, $\omega^{s',i}_{t,s}$, is implicitly determined as the level that ensures this equality. The investor buys contingent securities until the price-to-(perceived)probability ratio of a state (or the state price) is equal to the investor's relative marginal utility in that state.

Substituting (24) into (23) shows that investors allocate identical portfolio weights to the market portfolio, $\omega^{m,i}_{t,s} = \omega^m_{t,s}$. Intuitively, investors express their differences in beliefs through their holdings of contingent securities. Combining this observation with Eq. (17), we further obtain that, in equilibrium, these identical portfolio weights are equal to one,

$$\omega^{m,i}_{t,s} = 1 \text{ for each } i. \tag{25}$$

Output-asset price relation. We next show that there is a tight relationship between output and asset prices as in the two period model. Combining Eqs. (22) and (17) implies that aggregate consumption is a constant fraction of aggregate wealth,

$$\int_I c^i_{t,s} di = \rho Q_{t,s} k_{t,s}. \tag{26}$$

Combining this with Eq. (19), we obtain the output-asset price relation,

$$A\eta_{t,s} = \rho Q_{t,s}. \tag{27}$$

As before, full factor utilization, $\eta_{t,s} = 1$, obtains only if the price per unit of capital is at a particular level $Q^* \equiv A/\rho$. This is the efficient price level that ensures the implied consumption clears the goods market. Likewise, the economy features a demand recession, $\eta_{t,s} < 1$, if and only if the price per unit of capital is strictly below Q^*.

Using the output-asset price relation (and $y_{t,s} = A\eta_{t,s} k_{t,s}$), we can rewrite Eq. (16) as,

$$r^m_{t,s} = \rho + g + \mu^Q_{t,s}. \tag{28}$$

[11] The presence of state transitions makes the Sharpe ratio in our model slightly different than its common definition, which corresponds to the expected return in excess of the risk-free rate normalized by volatility.

In equilibrium, the dividend yield on the market portfolio is equal to the consumption rate ρ.

Combining the output-asset price relation with the interest rate policy in (21), we also summarize the goods market with,

$$Q_{t,s} \leq Q^*, r^f_{t,s} \geq 0, \text{ where at least one condition is an equality.} \tag{29}$$

In particular, the equilibrium at any time and state takes one of two forms. If the natural interest rate is nonnegative, then the interest rate policy ensures that the price per unit of capital is at the efficient level, $Q_{t,s} = Q^*$, capital is fully utilized, $\eta_{t,s} = 1$, and output is equal to its potential, $y_{t,s} = Ak_{t,s}$. Otherwise, the interest rate is constrained, $r^f_{t,s} = 0$, the price is at a lower level, $Q_{t,s} < Q^*$, and output is determined by aggregate demand according to Eq. (27).

For future reference, we also characterize the first-best equilibrium without interest rate rigidities. In this case, there is no lower bound constraint on the interest rate, so the price per unit of capital is at its efficient level at all times and states, $Q_{t,s} = Q^*$. Combining this with Eq. (28), we obtain

$$r^m_{t,s} = \rho + g. \tag{30}$$

Substituting this into Eq. (23) and using Eq. (25), we solve for "rstar" as,

$$r^{f*}_s = \rho + g - \sigma^2_s \text{ for each } s \in \{1, 2\}. \tag{31}$$

Hence, in the first-best equilibrium the risk premium shocks are fully absorbed by the interest rate. Next, we characterize the equilibrium with interest rate rigidities.

4. Common beliefs benchmark and amplification

In this section, we analyze the equilibrium in a benchmark case in which all investors share the same belief. That is, $\lambda^i_s \equiv \lambda_s$ for each i. We also normalize the total mass of investors to one so that individual and aggregate allocations are the same. We use this benchmark to illustrate how the spirals between asset prices and output exacerbate the recession, and how pessimism amplifies these spirals.

Because the model is linear, we conjecture that the price and the interest rate will remain constant within states, $Q_{t,s} = Q_s$ and $r^f_{t,s} = r^f_s$ (in particular, there is no price drift, $\mu^Q_{t,s} = 0$). Since the investors are identical, we also have $\omega^m_{t,s} = 1$ and $\omega^{s'}_{t,s} = 0$. In particular, the representative investor's wealth is equal to aggregate wealth, $a_{t,s} = Q_{t,s}k_{t,s}$. Combining this with Eq. (23) and substituting for $r^m_{t,s}$ from Eq. (28), we obtain the following risk balance conditions,

$$\sigma_s = \frac{\rho + g + \lambda_s \left(1 - \frac{Q_s}{Q_{s'}}\right) - r^f_s}{\sigma_s} \text{ for each } s \in \{1, 2\}. \tag{32}$$

These equations are the dynamic counterpart to Eq. (7) in the two-period model. They say that, in each state, the total risk in the economy (the left side) is equal to the Sharpe ratio perceived by the representative investor (the right side). Note that the Sharpe ratio accounts for the fact that the aggregate wealth (as well as the marginal utility) will change if there is a state transition.[12]

The equilibrium is then characterized by finding four unknowns, $\left(Q_1, r_1^f, Q_2, r_2^f\right)$, that solve the two equations (32) together with the two goods market equilibrium conditions (29). We solve these equations under the following parametric restriction.

Assumption 1. $\sigma_2^2 > \rho + g > \sigma_1^2$.

In view of this restriction, we conjecture an equilibrium in which the low-risk-premium state 1 features positive interest rates, efficient asset prices, and full factor utilization, $r_1^f > 0, Q_1 = Q^*$ and $\eta_1 = 1$, whereas the high-risk-premium state 2 features zero interest rates, lower asset prices, and imperfect factor utilization, $r_2^f = 0, Q_2 < Q^*$ and $\eta_2 < 1$. In particular, the analysis with common beliefs reduces to finding two unknowns, $\left(Q_2, r_1^f\right)$, that solve the two risk balance equations (32) (after substituting $Q_1 = Q^*$ and $r_2^f = 0$).

Equilibrium in the high-risk-premium state. After substituting $r_2^f = 0$, the risk balance equation (32) for the high-risk-premium state $s = 2$ can be written as,

$$\sigma_2 = \frac{\rho + g + \lambda_2 \left(1 - \frac{Q_2}{Q^*}\right)}{\sigma_2}. \tag{33}$$

In view of Assumption 1, if the price were at its efficient level, $Q_2 = Q^*$, the risk (the left side) would exceed the Sharpe ratio (the right side). As in the two period model, the economy generates too much risk relative to what the investors are willing to absorb at the constrained level of the interest rate. As before, the price per unit of capital, Q_2, needs to decline to equilibrate the risk markets. Rearranging the expression, we obtain a closed form solution,

$$Q_2 = Q^* \left(1 - \frac{\sigma_2^2 - (\rho + g)}{\lambda_2}\right). \tag{34}$$

As this expression illustrates, we require a minimum degree of optimism to ensure an equilibrium with positive price and output.

Assumption 2. $\lambda_2 > \sigma_2^2 - (\rho + g)$.

This requirement is a manifestation of an amplification mechanism that we describe next.

Amplification from endogenous output and earnings. In the two period model of Section 2, the future payoff from the market portfolio is exogenous (z_1). Therefore, a decline in the price

[12] To see this, observe that the term, $\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}$, in the equation is actually equal to, $\frac{Q_{t,s}}{Q_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}$. Here, $\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}$ denotes the capital gains and $\frac{Q_{t,s}}{Q_{t,s'}}$ denotes the marginal utility adjustment when there is a representative investor (see (23)).

of capital (Q) increases the dividend yield and the market return, $r^m(z_1) = z_1/Q$ [cf. Eq. (2)]. In contrast, in the current model the instantaneous payoff from the market portfolio is endogenous and given by $y_{t,2} = \rho Q_2 k_{t,2}$. Therefore, a decline in the price of the market portfolio does not affect the dividend yield ($\frac{y_{t,2}}{\rho Q_2 k_{t,2}} = \rho$) and leaves the market return absent transitions unchanged, $r^m = \rho + g$ [cf. Eq. (28)]. Unlike in the two period model, a decline in asset prices does not increase the market return any more (aside from state transitions). The intuition is that a lower price reduces output and economic activity, which reduces firms' earnings and leaves the dividend yield constant. Thus, asset price declines no longer play a stabilizing role, leaving the economy susceptible to a spiraling decline.

In view of this amplification mechanism, one might wonder how the risk market ever reaches equilibrium once the price, Q_2, starts to fall below its efficient level, Q^*. The stabilizing force is captured by the last term in Eq. (33), $\lambda_2\left(1 - \frac{Q_2}{Q^*}\right)$. A decline in the price increases the expected capital gain from transition into the recovery state $s = 1$, which increases the expected return to capital as well as the Sharpe ratio. The stabilizing force is stronger when investors are more optimistic and perceive a higher transition probability into the recovery state, λ_2. Assumption 2 ensures that the stabilizing force is sufficiently strong to counter the impact of the risk premium shock. If this assumption were violated, a risk premium shock would trigger a downward price spiral that would lead to an equilibrium with zero asset prices and zero output.

Finally, consider the comparative statics of the equilibrium price with respect to the exogenous shifter of the risk premium, σ_2^2 [cf. (31)]. Using Eq. (34), we obtain $\frac{d(Q_2/Q^*)}{d\sigma_2^2} = -\frac{1}{\lambda_2}$. Hence, risk premium shocks reduce asset prices (and output) by a greater magnitude when investors are more pessimistic about recovery (lower λ_2). These observations illustrate that beliefs matter in this environment not only because they have a direct impact on asset prices but also because they determine the strength of the amplification mechanism.

Equilibrium in the low-risk-premium state. Following similar steps for the low-risk-premium state $s = 1$, we also obtain a closed form solution for the interest rate in this state,

$$r_1^f = \rho + g - \sigma_1^2 - \lambda_1\left(\frac{Q^*}{Q_2} - 1\right). \tag{35}$$

Intuitively, given the expected return on capital, the interest rate adjusts to ensure that the risk balance condition is satisfied with the efficient price level, $Q_1 = Q^*$. For our conjectured equilibrium, we also assume an upper bound on λ_1 which ensures that the implied interest rate is positive.

Assumption 3. $\lambda_1 < \left(\rho + g - \sigma_1^2\right) / \left(Q^*/Q_2 - 1\right)$, where Q^*/Q_2 is given by Eq. (34).

Note also that Eq. (35) implies r_1^f is decreasing in the transition probability, λ_1, as well as in the asset price drop conditional on transition, Q^*/Q_2. Intuitively, interest rates are kept relatively low by the fact that investors fear a recession triggered by an increase in the risk premium and constrained interest rate (an endogenous "disaster").

The following result summarizes the characterization of equilibrium in this section. The testable predictions regarding the effect of risk premium shocks on consumption and output follow from combining the characterization with Eqs. (26) and (27).

Proposition 1. *Consider the model with two states, $s \in \{1,2\}$, with common beliefs and Assumptions 1-3. The low-risk-premium state 1 features a positive interest rate, efficient asset prices and full factor utilization, $r_1^f > 0, Q_1 = Q^*$ and $\eta_1 = 1$. The high-risk-premium state 2 features zero interest rate, lower asset prices, and a demand-driven recession, $r_2^f = 0, Q_2 < Q^*$, and $\eta_2 < 1$, as well as a lower level of consumption and output, $c_{t,2}/k_{t,2} = y_{t,2}/k_{t,2} = \rho Q_2$. The price in state 2 and the interest rate in state 1 are given by Eqs. (34) and (35).*

Equilibrium with investment and endogenous growth. In Appendix D.1, we extend the baseline environment to incorporate investment. This leads to two main changes. First, the growth rate in (14) becomes endogenous, $g_{t,s} = \varphi(\iota_{t,s}) - \delta$, where $\iota_{t,s} = \frac{i_{t,s}}{k_{t,s}}$ denotes investment rate per capital, $\varphi(\cdot)$ denotes a neoclassical production technology for capital, and δ denotes the depreciation rate. Second, under the simplifying assumption that output accrues to agents in the form of return to capital (i.e., no monopoly profits), optimal investment is an increasing function of the price per unit of capital, $Q_{t,s}$.[13] Moreover, using a convenient functional form for $\varphi(\cdot)$, we obtain a linear relation between the investment rate and the price, $\iota(Q_{t,s}) = \psi(Q_{t,s} - 1)$ for some $\psi > 0$.

In this setting, aggregate demand consists of the sum of consumption and investment. Using the expression for optimal investment, we also generalize the output-asset price relation (27) to,

$$A\eta_{t,s} = \rho Q_{t,s} + \psi(Q_{t,s} - 1). \tag{36}$$

Hence, output is increasing in asset prices not only because asset prices generate a wealth effect on consumption but also because they increase investment through a marginal-Q channel. Substituting optimal investment into the endogenous growth expression, we further obtain,

$$g_{t,s} = \psi q_{t,s} - \delta, \text{ where } q_{t,s} = \log Q_{t,s}. \tag{37}$$

Hence, this setting also features a *growth-asset price relation*: lower asset prices reduce investment, which translates into lower endogenous growth and lower potential output in future periods. The rest of the model is unchanged (see Appendix D.1 for details).

In Appendix D.2, we characterize the equilibrium in this extended environment and generalize Proposition 1. We find that risk premium shocks—captured by a transition to state 2—generate a decline in investment (and endogenous growth) as well as consumption and output as in the baseline version of the model. We test these predictions in Section 7. We also find that the decline in investment generates a second amplification mechanism that reinforces the mechanism we described earlier. Specifically, the recession lowers asset prices further not only by reducing

[13] Without this assumption, investment would be a function of $\tilde{Q}_{t,s} \leq Q_{t,s}$, which represents a claim on the rental rate of capital in future periods (excluding monopoly profits). The difference, $Q_{t,s} - \tilde{Q}_{t,s}$, captures the price of a claim on monopoly profits. Hence, allowing for profits would have a quantitative impact on investment, though we believe it would leave our qualitative results unchanged. We leave an investigation of this issue for future research.

output and earnings but also by reducing investment and growth (in potential output and earnings). Figure 1 in the introduction presents a graphical illustration of the two amplification mechanisms.

5. Belief disagreements and speculation

Going back to the baseline model, we next investigate the effect of belief disagreements. We show that *speculation* induced by belief disagreements exacerbates recessions and motivates macroprudential policy.

We restrict attention to two types of investors, optimists and pessimists, with beliefs denoted by, $\left\{\left(\lambda_1^i, \lambda_2^i\right)\right\}_{i \in \{o,p\}}$. We normalize the mass of each belief type to one so that $i = o$ and $i = p$ denote, respectively, the representative optimist and pessimist. We assume the beliefs satisfy the following:

Assumption 4. $\lambda_2^o > \lambda_2^p$ and $\lambda_1^o \leq \lambda_1^p$.

When the economy is in the high-risk-premium state, optimists find the transition into the low-risk-premium state relatively likely ($\lambda_2^o > \lambda_2^p$); when the economy is in the low-risk-premium state, optimists find the transition into the high-risk-premium state relatively unlikely ($\lambda_1^o \leq \lambda_1^p$). Hence, optimism and pessimism are *relative*: an optimist is someone who is optimistic relative to a pessimist. In fact, we do not need to specify the "objective distribution" for our theoretical results (including the welfare results). We do, however, need the relative optimism and pessimism to be persistent across the two risk premium states (see Remark 2 at the end of this section).

To characterize the equilibrium, we define the wealth-weighted average transition probability,

$$\overline{\lambda}_{t,s} \equiv \overline{\lambda}_s\left(\alpha_{t,s}\right) \equiv \alpha_{t,s}\lambda_s^o + \left(1-\alpha_{t,s}\right)\lambda_s^p, \text{ where } \alpha_{t,s}^o = \frac{a_{t,s}^o}{k_{t,s}Q_{t,s}}. \tag{38}$$

Here, $\alpha_{t,s}$ denotes optimists' wealth share, and it is the payoff-relevant state variable in this economy. The notation, $\overline{\lambda}_s\left(\alpha_{t,s}\right)$, describes the wealth-weighted average belief in state s as a function of optimists' wealth share, and $\overline{\lambda}_{t,s}$ denotes the belief at time t and state s. This belief is central to the analysis because the following analogue of the risk balance condition (32) holds in this setting (see Appendix B.3),

$$\sigma_s = \frac{1}{\sigma_s}\left(\rho + g + \mu_{t,s}^Q + \overline{\lambda}_{t,s}\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right) - r_{t,s}^f\right) \text{ for each } s \in \{1,2\}. \tag{39}$$

In particular, the equilibrium in risk markets is determined according to the wealth-weighted average belief. When $\alpha_{t,s}$ is greater, optimists exert a greater influence on asset prices. Note also that the expected return to the market portfolio features the price drift term, $\mu_{t,s}^Q$ [cf. (28)], which is not necessarily zero in this section because optimists' wealth share changes over time.

We must now characterize the dynamics of optimists' wealth share, $\alpha_{t,s}$ (and thus, the dynamics of $\overline{\lambda}_{t,s}$). Eq. (25) implies investors' weights on the market portfolio satisfy $\omega_{t,s}^{m,o} = \omega_{t,s}^{m,p} = 1$. In

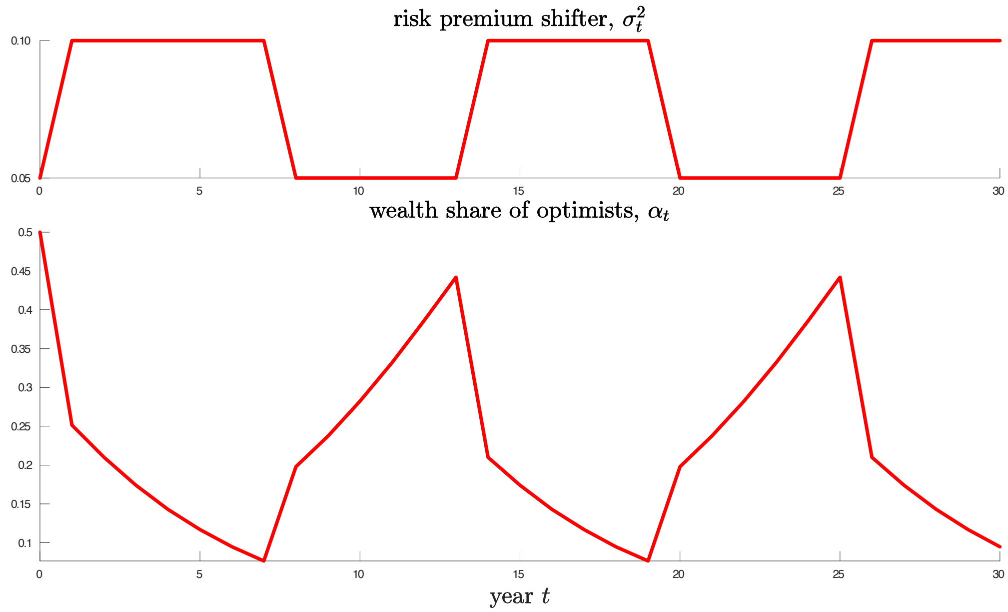

Figure 2: A simulation of the dynamics of optimists' wealth share over time.

Appendix B.3, we also solve for investors' weights on the contingent securities,

$$\omega_{t,s}^{s',o} = \lambda_s^o - \overline{\lambda}_{t,s} = (\lambda_s^o - \lambda_s^p)(1 - \alpha_{t,s}). \tag{40}$$

Thus, investors settle their disagreements on the jump risk by trading the contingent securities. Optimists take a positive position on a contingent security whenever their belief for the transition probability exceeds the weighted average belief. This implies that their wealth share evolves according to [cf. Eqs. ($B.13$) and ($B.14$)],

$$\begin{cases} \dot{\alpha}_{t,s} = (\lambda_s^p - \lambda_s^o)\,\alpha_{t,s}(1 - \alpha_{t,s}), & \text{if there is no state change,} \\ \alpha_{t,s'}/\alpha_{t,s} = \lambda_s^o/\overline{\lambda}_{t,s}, & \text{if there is a state change to } s'. \end{cases} \tag{41}$$

Here, $\dot{\alpha}_{t,s} = \frac{d\alpha_{t,s}}{dt}$ denotes the derivative with respect to time. As long as the economy remains in the boom state, optimists' wealth share drifts upwards (since $\lambda_1^o < \lambda_1^p$), because they make profits from selling insurance—contingent contracts that pay in the recession state. If there is a jump to the recession state, optimists' wealth share makes a downward jump. Conversely, optimists' wealth share drifts downwards in the recession state, and it makes an upward jump if there is a transition to the boom state. Figure 2 illustrates the dynamics of optimists' wealth share for a particular parameterization (described subsequently) and realization of uncertainty.

These observations also imply that the weighted average belief in (38) (that determines asset prices) is effectively *extrapolative* in the sense that good realizations increase effective optimism whereas bad realizations reduce it. Specifically, as the boom state persists, optimists' wealth share increases and the aggregate belief becomes more optimistic. After a transition to the recession state, the aggregate belief becomes less optimistic. Similarly, the aggregate belief becomes less optimistic as the recession persists, and it becomes more optimistic after a transition into the boom.

Eq. (41) determines the evolution of optimists' wealth share (and thus, the weighted average belief) regardless of the level of asset prices and output. The equilibrium is determined by jointly solving this expression together with the risk balance condition (39) and the goods market equilibrium condition (29). To make progress, we suppose Assumptions 1-3 from the previous section hold according to both belief types. This ensures that, regardless of the wealth shares, the low-risk-premium state 1 features a positive interest rate, efficient price level, and full factor utilization, $r^f_{t,1} > 0, Q_{t,1} = Q^*, \eta_{t,1} = 1$, and the high-risk-premium state 2 features a zero interest rate, a lower price level, and insufficient factor utilization, $r^f_{t,2} = 0, Q_{t,2} < Q^*, \eta_{t,2} < 1$. We next characterize this equilibrium starting with the high-risk-premium state. In this as well as the next section, we also find it convenient to work with the log of the price level, $q_{t,s} \equiv \log Q_{t,s}$.

Equilibrium in the high-risk-premium state. Consider the risk balance equation (39) for state $s = 2$. Using $\mu^Q_{t,2} = \frac{dQ_{t,2}/dt}{Q_{t,2}} = \dot{q}_{t,2}$, we obtain the following analogue of Eq. (33),

$$\sigma_2 = \frac{1}{\sigma_2}\left(\rho + g + \dot{q}_{t,2} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_2}{Q^*}\right)\right). \tag{42}$$

Combining this with Eq. (41), we obtain a differential equation system that describes the joint dynamics of the log price and optimists' wealth share, $(q_{t,2}, \alpha_{t,2})$, conditional on no transition. In Appendix B.3, we show that this system is saddle path stable: for any initial wealth share, $\alpha_{t,2} \in (0,1)$, there exists a unique equilibrium price level, $q_{t,2} \in [q^p_2, q^o_2)$, such that the solution satisfies $\lim_{t\to\infty} \alpha_{t,2} = 0$ and $\lim_{t\to\infty} q_{t,2} = q^p_2$. Here, q^i_2 denotes the log price level with common beliefs characterized in Section 4 corresponding to type i investors' belief. The system is also stationary, which implies that the price can be written as a function of optimists' wealth share. The price function, $q_2(\alpha)$, is characterized as the solution to the following differential equation in α-domain,

$$q_2'(\alpha)\left(\lambda^o_2 - \lambda^p_2\right)\alpha(1-\alpha) = \rho + g + \overline{\lambda}_2(\alpha)\left(1 - \frac{\exp(q_2(\alpha))}{Q^*}\right) - \sigma_2^2, \tag{43}$$

with boundary conditions, $q_2(0) = q^p_2$ and $q_2(1) = q^o_2$. We further show that $q_2(\alpha)$ is strictly increasing in α. As in the previous section, greater optimism increases the asset price in the high-risk-premium state.[14]

Equilibrium in the low-risk-premium state. Following similar steps for the risk balance condition for the low-risk-premium state $s = 1$, we obtain,

$$r^f_1(\alpha) = \rho + g - \overline{\lambda}_1(\alpha)\left(\frac{Q^*}{\exp(q_2(\alpha'))} - 1\right) - \sigma_1^2 \text{ where } \alpha' = \frac{\alpha\lambda^o_1}{\overline{\lambda}_1(\alpha)}. \tag{44}$$

[14] Introducing partial nominal price flexibility along the lines discussed in Remark 1 would create a second channel by which increasing optimists' wealth share would increase real asset prices. In that environment, pessimists would perceive lower expected inflation than optimists (because they believe the economy is more likely to stay in recession), which would lead to a greater perceived real interest rate and lower real asset valuations.

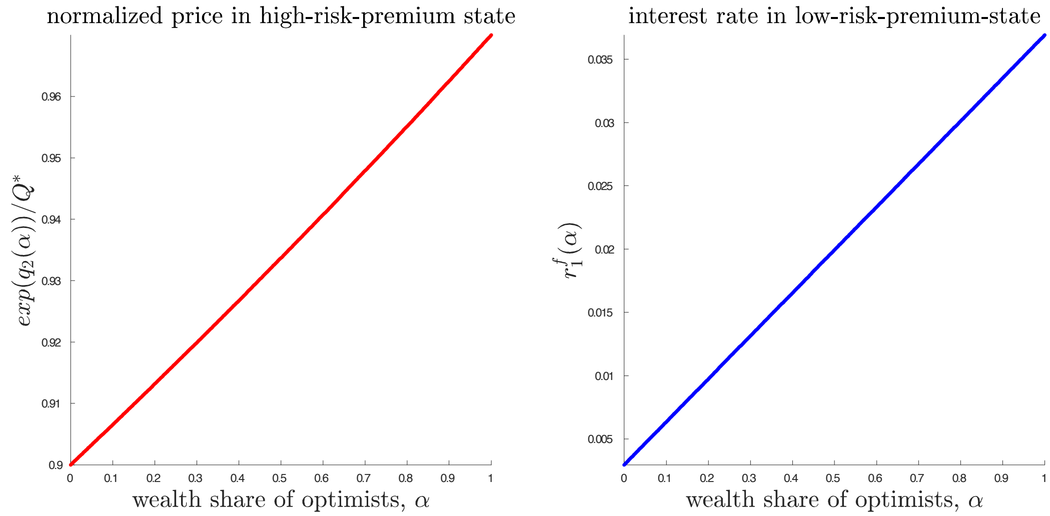

Figure 3: Equilibrium price and interest rate functions with heterogeneous beliefs.

Here, $r_1^f(\alpha)$ denotes the interest rate when optimists' wealth share is equal to α. The term, α', denotes optimists' wealth share after an immediate transition into the high-risk-premium state [cf. Eq. (41)]. The interest rate depends on (among other things) the weighted average transition probability into the high-risk-premium state, $\overline{\lambda}_1(\alpha)$, as well as the price level that would obtain after transition, $q_2(\alpha')$. It is easy to check that $r_1^f(\alpha)$ is increasing in α, since, as in the previous section, greater optimism increases asset prices.

The following proposition summarizes the characterization of equilibrium. The last part, which follows by combining the characterization with Eqs. (26) and (27), shows that greater optimists' wealth share in the high-risk-premium state mitigates the severity of the recession.

Proposition 2. *Consider the model with two belief types. Suppose Assumptions 1-3 hold for each belief, and that beliefs are ranked according to Assumption 4. Then, optimists' wealth share evolves according to Eq. (41). The equilibrium log-price and interest rate can be written as a function of optimists' wealth share,* $q_1(\alpha), r_1^f(\alpha), q_2(\alpha), r_2^f(\alpha)$. *In the low-risk-premium state,* $q_1(\alpha) = q^*$, *and* $r_1^f(\alpha)$ *is an increasing function of* α *given by Eq. (44). In the high-risk-premium state,* $r_2^f(\alpha) = 0$, *and* $q_2(\alpha)$ *is an increasing function of* α *that solves the differential equation (43) with* $q_2(0) = q_2^p$ *and* $q_2(1) = q_2^o$. *Greater optimists' wealth share in the high-risk-premium state,* $\alpha_{t,2}$, *increases the price per capital,* $Q_{t,2}$, *as well as consumption and output,* $c_{t,2}/k_{t,2} = y_{t,2}/k_{t,2} = \rho Q_{t,2}$.

Numerical illustration. We next illustrate the equilibrium using a simple parameterization (see Appendix B.4 for details). For the baseline parameters, we set $g = 5\%, \rho = 4\%, \sigma_1^2 = 5\%, \sigma_2^2 = 10\%$. For investors' beliefs about transition probabilities, we set $\lambda_1^o = 1/10, \lambda_1^p = 1/3$ for the boom state and $\lambda_2^o = 1/3, \lambda_2^p = 1/10$ for the recession state.

Figure 3 illustrates the corresponding equilibrium. The left panel illustrates the price of capital in the recession (normalized by the efficient price level) as a function of optimists' wealth share. When pessimists dominate the economy, the price of capital and output decline by 10%. In contrast,

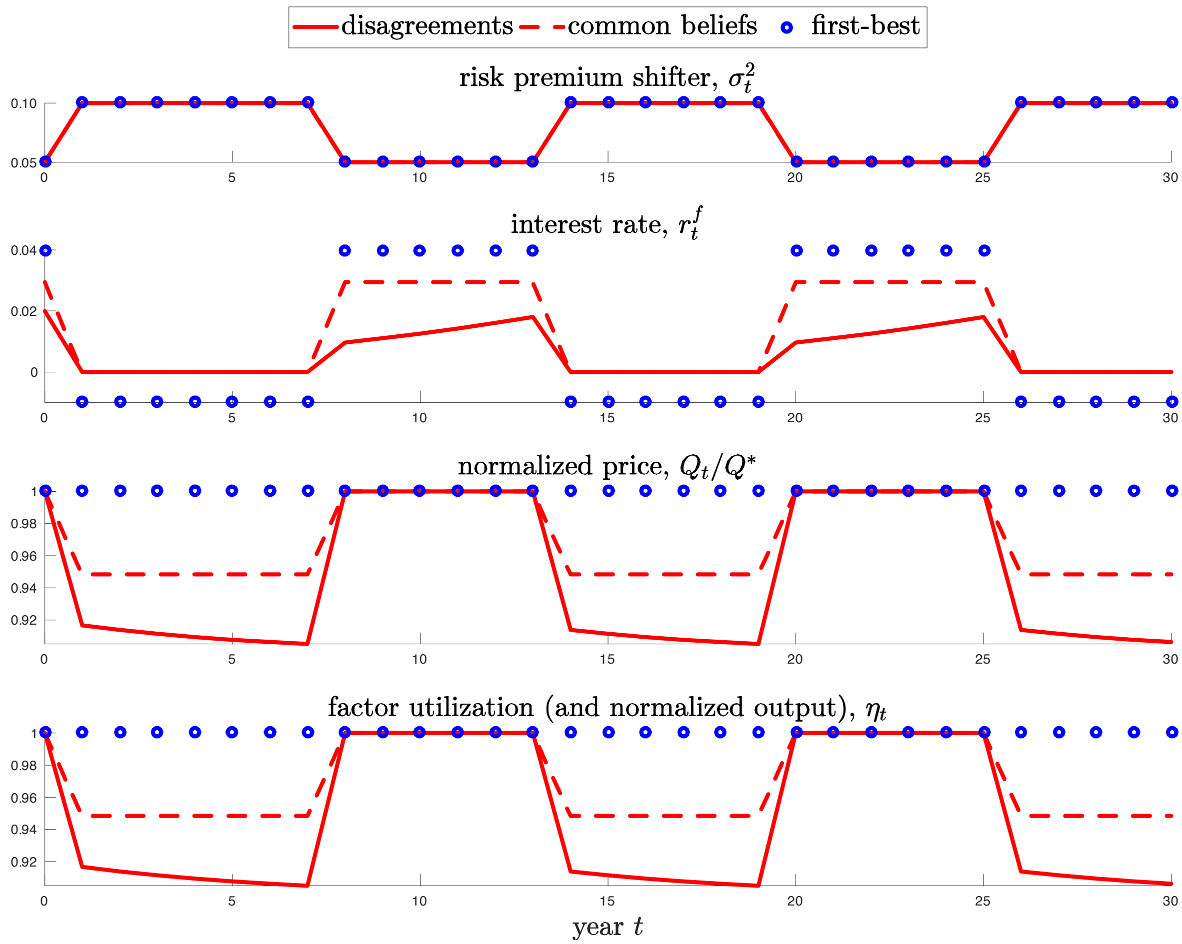

Figure 4: A simulation of the equilibrium variables over time with belief disagreements (solid red line), with common beliefs (dashed red line), and the first-best benchmark (circled blue line).

when optimists dominate, they decline by only 3%. The right panel of Figure 3 illustrates the interest rate in the boom as a function of optimists' wealth share. The risk-free rate during the boom is close to 4% when optimists dominate the economy but it is close to 0% when pessimists dominate.

Amplification from speculation. We next use our numerical example to illustrate how speculation further amplifies the business-cycle driven by risk premium shocks. To this end, we fix investors' beliefs and simulate the equilibrium for a particular realization of uncertainty over a 30-year horizon. We choose the (objective) simulation belief to be in the "middle" of optimists' and pessimists' beliefs in terms of the relative entropy distance.[15] Figure 4 illustrates the dynamics of equilibrium variables (except for optimists' wealth share, which we plot in Figure 2). For compari-

[15] This ensures that there is a non-degenerate long-run wealth distribution in which neither optimists nor pessimists permanently dominate, which helps to visualize the destabilizing effects of speculation without taking a stand on whether optimists and pessimists are "correct." Our welfare results in the next section do not require this assumption since we evaluate investors' expected utilities according to their own beliefs.

son, the dashed red line plots the equilibrium that would obtain in the common-beliefs benchmark if all investors shared the "middle" simulation belief, and the circled blue line plots the first-best equilibrium that would obtain without interest rate rigidities.

The figure illustrates two points. First, consistent with our baseline analysis in the previous section, the price per unit of capital is more volatile and the interest rate is more compressed than in the first-best equilibrium. In the high-risk-premium state, the interest rate cannot decline sufficiently to equilibrate the risk balance condition, which leads to a drop in asset prices and a demand recession. In the low-risk-premium state, the fear of transition into the recessionary high-risk-premium state keeps the interest rate lower than in the first-best benchmark.

Second, risk-centric recessions are more severe when investors have belief disagreements (and this also leads to more compressed interest rates). The intuition follows from Figures 2 and 3. Speculation in the low-risk-premium state decreases optimists' wealth share once the economy transitions into the high-risk-premium state, as illustrated by Figure 2, which translates into lower asset prices and a more severe demand recession, as illustrated by Figure 3 and Proposition 2. Speculation also increases optimists' wealth share if the boom continues, but this effect does not translate into higher asset prices or output since it is (optimally) neutralized by the interest rate response. The adverse effects of speculation on demand recessions motivates the analysis of macro-prudential policy, which we analyze in the next section.

Remark 2 (Interpretation of Belief Disagreements)**.** *As this discussion suggests, what matters for our results on speculation is* persistent heterogeneous valuations *for risky assets that ensure: (i) during the boom, high-valuation investors absorb relatively more of the recession risks, and (ii) during the recession, greater wealth share of high-valuation investors increases the (relative) price of risky assets. Belief disagreements generate these features naturally, under the mild assumption that optimists and pessimists do not flip roles across booms and recessions,*[16] *but other sources of heterogeneous valuations would lead to similar results. For example, with heterogeneity in risk aversion, more risk tolerant agents take on more aggregate risk (i.e., they insure less risk tolerant agents), which reduces their wealth share and the (relative) price of risky assets following negative shocks to fundamentals (see, for instance, Garleanu and Pedersen (2011); Longstaff and Wang (2012)). From this perspective, belief disagreements can also capture institutional reasons for heterogeneous valuations such as capacity or mandates for handling risk. Investment banks, for example, have far larger capacity to handle and lever risky positions than pensioners and money market funds.*

Formally, given two probability distributions $(p(\tilde{s}))_{\tilde{s}\in S}$ and $(q(\tilde{s}))_{\tilde{s}\in S}$, relative entropy of p with respect to q is defined as $\sum_{\tilde{s}} p(\tilde{s}) \log\left(\frac{p(\tilde{s})}{q(\tilde{s})}\right)$. Blume and Easley (2006) show that, in a setting with independent and identically distributed shocks (and identical discount factors), only investors whose beliefs have the maximal relative entropy distance to the true distribution survive. Since our setting features Markov shocks, we apply their result state-by-state to pick the simulation belief that ensures conditional transition probabilities satisfy the necessary survival condition for optimists as well as pessimists.

[16] This assumption is supported by recent survey evidence that shows belief heterogeneity is largely explained by persistent individual heterogeneity (Giglio et al. (2019)). The assumption is also consistent with an extensive psychology literature that documents the prevalence of optimism, as well as its heterogeneity and persistence, since it is largely a personal trait (see Carver et al. (2010) for a review).

6. Welfare analysis and macroprudential policy

Since our model features constrained monetary policy, most of the aggregate demand boosting policies that have been discussed in the New Keynesian literature are also effective in our environment. We skip a discussion of these policies for brevity (our results would still apply as long as these policies are imperfect). Instead, we focus on *macroprudential* policy interventions that impose restrictions on risk market participants with the objective of obtaining macroeconomic benefits. In practice, most macroprudential policies restrict risk taking by banks—especially large ones. Interpreting banks as relatively high-valuation investors (see Remark 2) or as lenders to such investors (see Section 7), we capture these policies in reduced form by imposing portfolio risk limits on relatively optimistic investors.

Our model features heterogeneous beliefs, which makes the welfare analysis challenging. We adopt the standard *Pareto* criterion in which the planner evaluates investors' expected utility according to their own beliefs. We adopt the standard criterion to highlight that our results are not driven by paternalistic concerns. Rather, the planner improves welfare by internalizing *aggregate demand externalities*. The standard criterion is also appropriate if we interpret belief disagreements as a modeling device to capture heterogeneous valuations due to other factors (see Remark 2). However, if we interpret belief disagreements literally, then a paternalistic criterion such as the belief-neutral welfare criterion developed by Brunnermeier et al. (2014) could be more appropriate. Adopting this belief-neutral criterion would reinforce our normative conclusions: in that case, macroprudential policy would not only improve macroeconomic outcomes but it would also mitigate the microeconomic costs associated with speculation (see, e.g., Simsek (2013b); Dávila (2017); Heimer and Simsek (2019)).

Using the standard welfare criterion also helps to simplify the theoretical analysis. Since our model features complete markets and no frictions other than interest rate rigidities, aggregate demand externalities constitute the only source of inefficiency. Therefore, the first-best benchmark that also corrects for these inefficiencies is Pareto efficient. This enables us to isolate the aggregate demand externalities by defining investors' *gap values*: the difference between their expected value in equilibrium relative to the expected value in the first-best benchmark. The gap value captures the present discounted value of the investor's utility losses due to demand recessions. Focusing on the gap value simplifies the analysis considerably because, up to a first order, macroprudential policies affect social welfare only through their impact on investors' gap values (in view of the fact that the first-best benchmark is Pareto efficient).

The formal analysis in this section proceeds as follows. Using the model with two belief types from the previous section, we first characterize investors' value functions in equilibrium according to their own beliefs. We define the gap value functions and illustrate the aggregate demand externalities. We then show that macroprudential policy that induces optimists to act more pessimistically (via appropriate portfolio risk limits), but that otherwise does not distort allocations, can generate a *Pareto* improvement of social welfare. We focus on macroprudential policy in the boom (low-risk-premium) state and provide a brief discussion of the macroprudential policy in the recession (high-risk-premium) state.

Value function in equilibrium. Because the model is linear, investors' expected utility can be written as (see Appendix B.1.1),

$$V^i_{t,s}\left(a^i_{t,s}\right) = \frac{\log\left(a^i_{t,s}/Q_{t,s}\right)}{\rho} + v^i_{t,s}. \tag{45}$$

Here, $v^i_{t,s}$ denotes the normalized value function per unit of capital stock. In Appendix C.1, we further characterize it as the solution to the following differential equation system,

$$\rho v^i_{t,s} - \frac{\partial v^i_{t,s}}{\partial t} = \log\rho + q_{t,s} + \frac{1}{\rho}\left(\begin{array}{c} g - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^i_s}{\overline{\lambda}_{t,s}}\right)\end{array}\right) + \lambda^i_s\left(v^i_{t,s'} - v^i_{t,s}\right). \tag{46}$$

The equilibrium price, $q_{t,s}$, affects investors' welfare since it determines output and consumption [cf. Eqs. (26) and (27)]. Consumption growth, g, and volatility, σ_s^2, also affect welfare. Finally, speculation affects investors' (perceived) welfare. This is captured by the term, $-\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^i_s}{\overline{\lambda}_{t,s}}\right)$, which is zero with common beliefs, and strictly positive with disagreements.

Gap value function. To facilitate the policy analysis, we break down the value function into two components,

$$v^i_{t,s} = v^{i,*}_{t,s} + w^i_{t,s}. \tag{47}$$

Here, $v^{i,*}_{t,s}$ denotes *the first-best value function* that would obtain if there were no interest rate rigidities. It is characterized by solving Eq. (46) with the efficient price level, $q_{t,s} = q^*$, for each t, s. The residual, $w^i_{t,s} = v^i_{t,s} - v^{i,*}_{t,s}$, denotes *the gap value function*, which captures the loss of value due to interest rate rigidities and demand recessions. As we will see below, the first-order impact of macroprudential policy on social welfare depends only on the gap value function. Using Eq. (46), we characterize the gap value function as the solution to the following system,

$$\rho w^i_{t,s} = q_{t,s} - q^* + \frac{\partial w^i_{t,s}}{\partial t} + \lambda^i_s\left(w^i_{t,s'} - w^i_{t,s}\right). \tag{48}$$

This illustrates that, in view of the output-asset price relation (27), the gap value function depends on the asset prices relative to the efficient level. Recall also that the equilibrium features $q_{t,1} = q^*$ and $q_{t,2} < q^*$. Thus, the key objective of policy interventions in this environment is to increase the asset price in the high-risk-premium state (so as to mitigate the demand recession).

Aggregate demand externalities. In Appendix C.1, we show that the gap value function can be written as a function of optimists' wealth share, $w^i_s(\alpha)$. Combining Eqs. (48) and (41), we also characterize this function as the solution to the following system in α-domain,

$$\rho w_s^i(\alpha) = q_s(\alpha) - q^* - (\lambda_s^o - \lambda_s^p)\,\alpha(1-\alpha)\frac{\partial w_s^i(\alpha)}{\partial \alpha} + \lambda_s^i\left(w_{s'}^i(\alpha') - w_s^i(\alpha)\right), \tag{49}$$

where $\alpha' = \alpha\lambda_s^o/\overline{\lambda}_s(\alpha)$. Recall that the price function in the high-risk-premium state, $q_2(\alpha)$, is increasing in optimists' wealth share [cf. Figure 3]. This leads to the following result.

Lemma 1. *The gap value function satisfies,* $\frac{dw_s^i(\alpha)}{d\alpha} > 0$ *for each* s, i *and* $\alpha \in (0,1)$.

Intuitively, optimists' wealth share is a scarce resource that brings asset prices and output in the high-risk-premium state closer to its first-best level. Thus, the gap value function in the high-risk-premium state is increasing in optimists' wealth share. The gap value function in the other state is also increasing, because the economy can always transition into the high-risk-premium state, where optimists' wealth share is useful (see Lemma 2 below for a ranking of the marginal value of optimists' wealth share across the two states).

The result also illustrates the *aggregate demand externalities*. Optimists' wealth share is an endogenous variable that fluctuates due to investors' portfolio decisions [cf. Figure 2]. Individual optimists that take positions in contingent markets—and pessimists that take the other side of these positions—do not take into account the impact of their decisions on asset prices and social welfare. This leads to inefficiencies that can be corrected by macroprudential policy.

Equilibrium and gap value functions with macroprudential policy. To evaluate the direction of the inefficiency, we consider a constrained policy exercise where the planner can induce optimists to choose allocations as if they have less optimistic beliefs.[17] Specifically, optimists are constrained to choose allocations *as-if* they have the beliefs, $\lambda^{o,pl} \equiv \left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, that satisfy, $\lambda_1^{o,pl} \geq \lambda_1^o$ and $\lambda_2^{o,pl} \leq \lambda_2^o$. Pessimists continue to choose allocations according to their own beliefs. Throughout, we use $\lambda_s^{i,pl}$ to denote investors' as-if beliefs and λ_s^i to denote their actual beliefs (for pessimists, the two beliefs coincide). We also use $\overline{\lambda}_s^{pl}(\alpha) = \alpha\lambda_s^{o,pl} + (1-\alpha)\lambda_s^p$ to denote the weighted average as-if belief.

In Appendix C.2, we show that the planner can implement this policy by imposing inequality restrictions on optimists' portfolio weights, while allowing them to make unconstrained consumption-savings decisions. Specifically, when the risk premium is low, the policy constrains optimists from taking too negative a position on the contingent security that pays if there is a transition to the high-risk-premium state, $\omega_{t,1}^{2,o} \geq \underline{\omega}_{t,1}^{2,o}$ (restrictions on selling "put options"). When the risk premium is high, the policy constrains optimists from taking too large a position on the contingent security that pays if there is a transition to the low-risk-premium state, $\omega_{t,2}^{1,o} \leq \overline{\omega}_{t,2}^{1,o}$ (restrictions on buying "call options"). Finally, in either state, the policy also constrains optimists' weight on the market portfolio not to exceed the market average, $\omega_{t,s}^{m,o} \leq 1$ (since otherwise optimists start to speculate by increasing their exposure to the market portfolio).

[17] For simplicity, we restrict attention to time-invariant policies. The planner commits to a policy at time zero, $\left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, and implements it throughout.

The characterization of equilibrium with policy is then the same as in Section 5. In particular, Eqs. (41) and (42) still hold with the only difference that investors' beliefs are replaced with their as-if beliefs, $\lambda_s^{i,pl}$. We denote the resulting price functions with $q_s^{pl}(\alpha)$ to emphasize that they are determined by as-if beliefs (as opposed to actual beliefs). On the other hand, the equation system that characterizes the gap value function is given by,

$$\rho w_s^i(\alpha) = q_s^{pl}(\alpha) - q^* - \left(\lambda_s^{o,pl} - \lambda_s^p\right)\alpha(1-\alpha)\frac{\partial w_s^i(\alpha)}{\partial \alpha} + \lambda_s^i\left(w_{s'}^i\left(\alpha'^{,pl}\right) - w_s^i(\alpha)\right) \tag{50}$$

where $\alpha'^{,pl} = \alpha\lambda_s^{o,pl}/\overline{\lambda}_s^{o,pl}(\alpha)$. Comparing this with Eq. (49) illustrates that the macroprudential policy can affect the gap value through two potential channels. First, it might affect the equilibrium asset prices (captured by the term, $q_s^{pl}(\alpha)$). Second, the policy affects the dynamics of optimists' wealth share, which in turn influence the gap value. For example, in the low-risk-premium state $s=1$, the policy increases $\lambda_1^{o,pl}$, which induces optimists to increase their position on the contingent security that pays if there is a transition into the high-risk-premium state [cf. Eq. (40)]. This increases optimists' wealth share after a transition (captured by the term, $\alpha'^{,pl}$) at the expense of reducing optimists' wealth share in case there is no transition (captured by the term, $-\left(\lambda_s^{o,pl} - \lambda_s^p\right)$).

Planner's Pareto problem. To trace the Pareto frontier, we allow the planner to make a one-time wealth transfer among the investors at time zero. In Appendix C.2, we show that the planner's Pareto problem can then be reduced to,

$$\max_{\lambda^{o,pl}} v_{0,s}^{pl} = \alpha_{0,s}v_{0,s}^o + (1-\alpha_{0,s})\,v_{0,s}^p. \tag{51}$$

Hence, the planner maximizes a wealth-weighted average of investors' normalized values (where the wealth shares correspond to Pareto weights). We also decompose the planner's value function into first-best and gap value components, $v_{0,s}^{pl} = v_{0,s}^{pl,*} + w_{0,s}^{pl}$.

A key observation is that, since the first-best benchmark does not feature any frictions, it satisfies the First Welfare Theorem and therefore it is Pareto efficient. This in turn implies that the marginal impact of the policy on the planner's first-best value function is zero, $\left.\frac{\partial v_{0,s}^{pl,*}}{\partial \lambda^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^o} = 0.$ [18] Consequently, the first order impact of the policy is characterized by its impact on the planner's gap value function,

$$w_{0,s}^{pl} = \alpha_{0,s}w_{0,s}^o + (1-\alpha_{0,s})\,w_{0,s}^p. \tag{52}$$

Macroprudential policy in the low-risk-premium state. Now suppose the economy is in the low-risk-premium state $s=1$. The planner can use macroprudential policy in the current state, $\lambda_1^{o,pl} \geq \lambda_1^o$ (she can induce optimists to act as if transition into the recession is more likely), but not in the other state $\lambda_2^{o,pl} = \lambda_2^o$ (she cannot influence optimists' actions in the recession state).

[18] If this wasn't the case, the first-best allocations could be Pareto improved by appropriately changing optimists' as-if beliefs.

Effectively, this policy induces optimists to sell less of the contingent security that pays in case there is a transition to the high-risk-premium state, while also preventing optimists from increasing their position in the market portfolio.

For small changes, this policy does not affect the price function in the current state, $q_1^{pl}(\alpha) = q^*$ (since we assume beliefs in the boom state are sufficiently optimistic that the interest rate is not constrained—see Assumption 3). Hence, the policy affects the gap value only through its impact on optimists' wealth dynamics and the associated aggregate demand externalities. Differentiating Eq. (50) (for $s = 1$) with respect to optimists' as-if beliefs and evaluating at the no-policy benchmark ($\lambda_1^{o,pl} = \lambda_1^o$), we obtain,

$$\left(\rho + \lambda_1^i\right) \frac{\partial w_1^i(\alpha)}{\partial \lambda_1^{o,pl}} = \alpha(1-\alpha)\left[-\frac{\partial w_1^i(\alpha)}{\partial \alpha} + \frac{\lambda_1^i}{\overline{\lambda}_1(\alpha)} \frac{\lambda_1^p}{\overline{\lambda}_1(\alpha)} \frac{\partial w_2^i(\alpha')}{\partial \alpha}\right] + \lambda_1^i \frac{\partial w_2^i(\alpha)}{\partial \lambda_1^{o,pl}}, \tag{53}$$

where $\alpha' = \alpha\lambda_1^o/\overline{\lambda}_1(\alpha)$. Here, the two terms inside the brackets capture the direct impact of the policy on welfare through aggregate demand externalities. The second term illustrates that the policy generates positive aggregate demand externalities—because it increases optimists' wealth share if there is a transition into the high-risk-premium state. On the other hand, the first term illustrates that the policy also generates negative aggregate demand externalities—because it reduces optimists' wealth share in case there is no transition. Eq. (53) describes the balance of these externalities when optimists are required to purchase the contingent security at equilibrium prices.

This illustrates that, in a dynamic setting, macroprudential policy in the low-risk-premium state is associated with some costs as well as benefits. The costs emerge from the fact that the policy prevents optimists from accumulating wealth that could be useful in a *future recession*. However, intuition suggests the benefits should outweigh the costs as long as future recessions are not too different from an imminent recession. The following lemma verifies this for the special case, $\lambda_1^o = \lambda_1^p$.

Lemma 2. *When $\lambda_1^o = \lambda_1^p$, the gap value function satisfies $\frac{dw_2^i(\alpha')}{d\alpha} > \frac{dw_1^i(\alpha)}{d\alpha}$ for each i and $\alpha \in (0,1)$.*

That is, optimists' wealth share increases the gap value more when there is an immediate transition into the high-risk-premium state, in which case the benefits appear immediately. Any delay in such transition reduces the benefits by postponing them. Combining this lemma with Eq. (53) provides a heuristic derivation of our main result in this section (see Appendix C.2 for the proof).

Proposition 3. *Consider the model with two belief types that satisfy $\lambda_1^o = \lambda_1^p$. Consider the macroprudential policy in the boom state, $\lambda_1^{o,pl} \geq \lambda_1^o$ (and suppose $\lambda_2^{o,pl} = \lambda_2^o$). The policy increases the planner's gap value (and thus, also the total value),*

$$\left.\frac{\partial v_1^{pl}(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda_1^{o,pl}=\lambda_1^o} = \left.\frac{\partial w_1^{pl}(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda_1^{o,pl}=\lambda_1^o} > 0 \text{ for each } \alpha \in (0,1).$$

In particular, regardless of the planner's Pareto weight, there exists a Pareto improving macroprudential policy.

What happens when we relax the assumption, $\lambda_1^o = \lambda_1^p$? This is largely a technical assumption. We conjecture that Proposition 3 also holds when $\lambda_1^o < \lambda_1^p$ (under appropriate technical assumptions) but we are unable to provide a proof. There are two distinct challenges. First, we cannot generalize Lemma 2, although the ranking is intuitive and should hold unless there are strong nonlinearities in the gap value function.[19] Second, in the more general case pessimists and optimists disagree about the benefits of macroprudential policy (captured by λ_1^i in the bracketed terms of (53)). The planner takes a weighted average of these perceptions, which complicates the analysis.[20]

Figure 5 illustrates the result for our earlier parameterization (that features $\lambda_1^o < \lambda_1^p$). We fix the optimists' wealth share at a particular level ($\alpha = \frac{1}{2}$) and calculate the effect of macroprudential policy on the planner's value function as well as on its components. The policy reduces the planner's first-best value function, since it distorts investors' allocations according to their own beliefs. However, for small policy changes, the magnitude of this decline is small (due to the First Welfare Theorem). The policy also generates a relatively sizeable increase in the planner's gap value function. This increase is sufficiently large that the policy increases the actual value function and generates a Pareto improvement. As the policy becomes larger, the gap value continues to increase whereas the first-best value decreases. Moreover, the decline in the first-best value is negligible for small policy changes but it becomes sizeable for large policy changes. The (constrained) optimal macroprudential policy obtains at an intermediate level.

The result is reminiscent of the analysis in Korinek and Simsek (2016), in which macroprudential policy improves outcomes by increasing the wealth of high marginal propensity to consume (MPC) households when there is a demand-driven recession. While both results are driven by aggregate demand externalities, the mechanism here is different and operates via asset prices. In fact, in our setting, all investors have the same MPC equal to ρ. Optimists improve aggregate demand in the high-risk-premium state not because they spend more than pessimists, but because they increase asset prices and induce *all investors* to spend more.

Macroprudential policy in the high-risk-premium state. The analysis so far concerns macroprudential policy in the low-risk-premium state and maintains the assumption that $\lambda_2^{o,pl} = \lambda_2^o$. In Appendix C.2, we also analyze the polar opposite case when the economy is in the high-risk-premium state $s = 2$, and the planner can apply macroprudential policy in this state, $\lambda_2^{o,pl} \leq \lambda_2^o$ (she

[19] Specifically, the proof of Lemma 2 establishes,

$$\frac{\partial w_1^i\left(b_{0,1}\right)}{\partial b} = \frac{\lambda_1^i}{\lambda_1^i + \rho} \int_0^\infty e^{-\left(\rho + \lambda_1^i\right)t} \left(\rho + \lambda_1^i\right) \frac{\partial w_2^i\left(b_{t,2}\right)}{\partial b} dt,$$

where $b_{0,1}$ denotes a transformed version of α at the initial state, and $b_{t,2}$ denotes the same variable after a transition into the high-risk-premium state after a period of length t. When $\lambda_1^o = \lambda_1^p$, we also have $b_{t,2} = b_{0,1}$ (since there is no speculation in the low-risk state), which yields $\frac{\partial w_1^i(b_{0,1})}{\partial b} = \frac{\lambda_1^i}{\lambda_1^i + \rho} \frac{\partial w_2^i(b_{0,1})}{\partial b} < \frac{\partial w_2^i(b_{0,1})}{\partial b}$. When $\lambda_1^o < \lambda_1^p$, the same result holds and the ranking remains unchanged if the value function is linear in the transformed variable b. Hence, the ranking can fail only if there are sufficiently large nonlinearities in the gap value function.

[20] When $\lambda_1^o = \lambda_1^p$, we actually have the stronger result that $\frac{\partial w_1^i(\alpha)}{\partial \lambda_1^{o,pl}} > 0$ for each i, that is, the policy increases the gap value according to optimists and pessimists (see Eq. ($C.18$)). We state the weaker version of the result in Proposition 3 because the stronger version might conceivably fail according to optimists (e.g., if λ_1^o is close to zero).

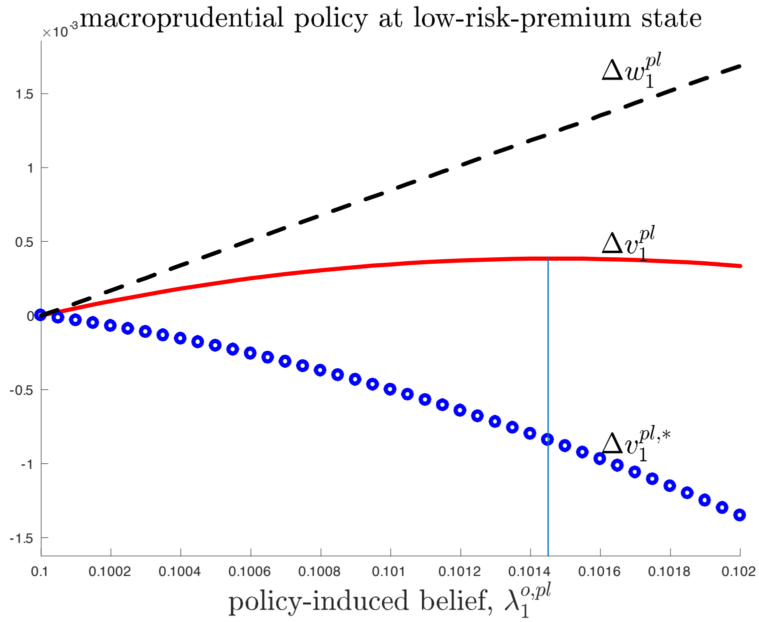

Figure 5: Effect of macroprudential policy in the low-risk-premium state on the planner's value function and its components.

can induce optimists to act as if the recovery is less likely), but not in the other state, $\lambda_1^{o,pl} = \lambda_1^o$. Proposition 4 in the appendix shows that, in contrast to Proposition 3, this policy can reduce social welfare. Consider the two counteracting forces. First, similar to before, macroprudential policy increases the gap value by increasing optimists' wealth share if the economy stays at the high-risk-premium state. However, unlike before, macroprudential policy also reduces current asset prices because the price is below the first-best level, $q_2^{pl}(\alpha) < q^*$, and it is increasing in optimists' as-if optimism, $\lambda_2^{o,pl}$ (see Eq. (34)). This channel reduces the gap value (see Eq. (50)). When optimists' wealth share is large ($\alpha \to 1$), the latter channel is dominant and macroprudential policy reduces the gap value and the social welfare. Even when the latter channel does not dominate, it suggests that the macroprudential policy in the recession state is less useful than in the boom state (which we verify in numerical simulations).

It is useful to emphasize that macroprudential policy in the low-risk-premium state does not lower asset prices due to the monetary policy response. Specifically, while the asset price in this state is not influenced by policy, $q_1^{pl}(\alpha) = q^*$, the interest rate, $r_1^f(\alpha)$, is decreasing in optimists' as-if pessimism, $\lambda_1^{o,pl}$ (see Eq. (35)). Intuitively, as macroprudential policy reduces the demand for risky assets, monetary policy lowers the interest rate to dampen its effect on asset prices and aggregate demand.

Taken together, our analysis provides support for *procyclical* macroprudential policy. In states where output is not demand constrained (in our model, the boom state $s = 1$), macroprudential policy that restricts high-valuation investors' (in our model optimists') risk taking is desirable. This policy improves welfare by ensuring that high-valuation investors bring more wealth to the demand-constrained states, which increases asset prices and output. In states where output is demand constrained (in our model, the recession state $s = 2$), macroprudential policy is less useful because it has an immediate negative impact on asset prices and aggregate demand.

7. Empirical evidence

Our empirical analysis focuses on three predictions. First, our model predicts that risk premium shocks generate an interest rate reduction when the interest rate is not constrained, and a more severe demand recession when the interest rate is constrained. Second, the recession reduces firms' earnings and leads to a further reduction in asset prices. Third, the recession is more severe when the shock takes place in an environment with more speculation. To investigate these predictions, we compare the response to house price shocks in Eurozone countries (which have constrained interest rates with respect to national shocks) to the response in non-Eurozone developed countries (which have less constrained interest rates). At the end of the section, we discuss empirical evidence from the recent literature which suggests that similar results apply for price shocks to other asset classes, such as stocks, as well as for other constraints on the interest rate, such as the zero lower bound.

While our model relies on the zero lower bound constraint, the mechanisms are more general, and we find it more convenient to work with the currency-union constraint in our empirical analysis. The zero lower bound has only recently become a practical constraint, generating data limitations, and it calls for an asymmetric specification that requires separate responses to positive and negative price shocks (since the monetary policy can raise the interest rate in response to positive shocks, especially if the economy is close to full capacity utilization). In contrast, individual Eurozone countries have had constrained interest rates (with respect to national shocks) for much longer, and the constraint has been symmetric with respect to the direction of shocks.

A major challenge in this exercise is the identification of the risk premium shock that drives asset prices. As we clarify in Section 2, the exact source of the shock is not important for our mechanisms (e.g., risk, risk aversion, or beliefs have similar effects). Therefore, our strategy is to control for factors that *do not* act as a risk premium shock according to our model. In particular, we attempt to control for supply shocks and demand shocks that are not specific to house prices—including monetary policy shocks, and we interpret the residual change in house prices as a plausibly exogenous risk premium shock. Specifically, our risk premium shock is a surprise change in house prices in a country after controlling for contemporaneous and recent changes in output and the policy interest rate (as well as the average house price change in sample countries).[21]

Our model has a single type of capital, which can be interpreted as a value-weighted average of housing, stocks, and other assets. We focus on house prices for two reasons. First, housing wealth is large and its size (relative to output) is comparable between Eurozone and non-Eurozone developed countries (see Table 3 in Appendix E). In contrast, stock markets in Eurozone countries

[21] While our controls are imperfect, we also report the differential effects of these shocks in Eurozone countries compared to their effects outside the Eurozone, which provides additional robustness. For example, our model illustrates that permanent supply shocks (e.g., an increase in A) shift asset prices and output regardless of whether the interest rate is constrained (see Sections 3 and 4). This suggests that common omitted supply shocks would lead to a similar bias inside and outside the Eurozone that is mitigated by focusing on the differential responses.

are typically much smaller than in non-Eurozone developed countries, which makes stocks less suitable for our empirical strategy (see Table 4). Second, house prices are less volatile and seem to react to monetary policy shocks with some delay (see Figure 15 in Appendix E). This feature enables us to control for monetary policy shocks by including contemporaneous and past realization of policy interest rates. We also interpret the future changes in interest rates as the monetary policy response to the risk premium shock, which enables us to test a key prediction of our model. This strategy works less well for stocks, because stock prices react to monetary policy shocks quickly, which might create a correlation between stock prices and interest rates with the opposite sign (since stock price declines driven by monetary policy shocks are typically followed by interest rate *hikes*—the opposite of risk premium shocks).[22]

Data sources. We assemble a quarterly cross-country panel data set of financial and economic variables for advanced economies. We obtain data on house price indices from the quarterly dataset described in Mack et al. (2011). We obtain data on macroeconomic activity such as GDP, investment, and consumption from the OECD. We also obtain financial market data such as the policy interest rate, stock price indices, and earnings (of publicly traded firms) from Global Financial Data (GFD) and the Bank for International Settlements (BIS). Appendix E describes the details of data sources and variable construction.

Sample selection. Our sample covers 21 advanced economies from the first quarter of 1990 until the last quarter of 2017. Our selection of countries is driven by the availability of consistent house price data. We start the sample in 1990 because monetary policy in most advanced economies had shifted from focusing on stabilizing inflation to stabilizing output by this time, as in our model. Our results are robust to alternative sample selections.[23]

To capture interest rate constraints, we divide the data into two categories. The first category, which we refer to as *the Euro/ERM sample*, consists of country-quarters in which the country was a member of the Euro area or the European Exchange Rate Mechanism (ERM) for most of the calendar year. The ERM system, which was introduced as a precedent to the Euro, requires the member countries to keep their exchange rates within a narrow band of a central currency. This system constrains countries' relative policy interest rates (albeit imperfectly) and most member countries eventually adopted the Euro. The countries in the Euro area share the same policy interest rate (determined by the European Central Bank). The second category, which we refer to as *the non-Euro/ERM sample*, consists of the remaining country-quarters. Table 1 in Appendix E describes the Euro/ERM status by country and year.

[22] Formally, we assume house prices react to monetary policy shocks with a delay of at least one quarter. Figure 15 in the appendix plots impulse responses to shocks to the policy interest rate and provides support for this assumption. Specifically, a surprise increase in the policy interest rate is followed by a decline in house prices, but the response starts after the first quarter and takes several quarters to complete. The same figure also shows that the assumption is clearly violated for stock prices. A surprise increase in the policy interest rate also reduces stock prices, but all of the response takes place in the same quarter as the shock.

[23] Figures 13 and 14 in the appendix show that starting the sample in 1980 leaves our results (except for the effect on inflation) qualitatively unchanged.

Empirical specification. To describe how the economy behaves after house price shocks, we follow the local projection method developed by Jordà (2005). In particular, we regress several outcome variables at various horizons after time t on (residual) house price changes at time t. Specifically, we estimate equations of the type,

$$Y_{j,t+h}^{h} - Y_{j,t-1}^{h} = \alpha_j^h + \gamma_t^h + \beta^{p,h}\left(-\Delta \log P_{j,t}\right) + \beta^{c,h}\text{controls}_{j,t} + \varepsilon_{j,t}^{h}, \tag{54}$$

where j denotes the country, t denotes the quarter, h denotes the horizon, Y denotes an outcome variable, P denotes the (real) house price index, and $\Delta \log P_{j,t} = \log P_{j,t} - \log P_{j,t-1}$ denotes its quarterly log change. We include time as well as country fixed effects so our "house price shock" is a decline in house prices in a quarter, after accounting for the average price decline in the sample countries as well as various other controls within the country. Our control variables include the contemporaneous value and 12 lags of the first difference of log GDP—to control for supply shocks and demand shocks that are not specific to house prices. Likewise, we include the contemporaneous value and 12 lags of the policy interest rate—to control for monetary policy shocks. We also include 12 lags of the first difference of log house prices—to capture the momentum in house prices, and 12 lags of the first difference of the outcome variable—to control for other dynamics that might influence the outcomes. We weight each regression with countries' relative GDP, and estimate (54) for horizons 0 to 12.

To evaluate the responses within and outside the Eurozone, we also include indicator variables for Euro/ERM and non-Euro/ERM status, and we interact all right-hand-side variables (including the fixed effects) with these indicators. We let $\beta_{euro}^{p,h}$ and $\beta_{non}^{p,h}$ denote the coefficient on the interaction of the price shock with the corresponding indicator. Our specification is equivalent to running the regressions separately within the Euro/ERM and non-Euro/ERM samples.[24] We report the sequence of coefficients, $\left\{\beta_{euro}^{p,h}\right\}_{h=0}^{12}$ and $\left\{\beta_{non}^{p,h}\right\}_{h=0}^{12}$, which provide an estimate of the impulse response functions for the respective samples. We also report 95% confidence intervals calculated according to Newey and West (1987) standard errors with a bandwidth of 20 quarters.

Our outcome variables include terms for which our model makes a clear prediction, such as the policy interest rate, the unemployment rate (a proxy for factor underutilization), the logs of GDP, investment, and consumption. We also include the log (core) CPI. Even though it is constant in our model (by assumption), variants of our model predict that it should decline in a demand recession. We also analyze public firms' earnings and log stock prices to investigate spillover and amplification effects, as well as log house prices to investigate the price dynamics following the initial shock. All relevant variables except for the policy interest rate are adjusted for inflation to focus on real effects, as in our model. For earnings, we use the ratio of earnings to the initial stock

[24] The point estimates from our regression are identical to those obtained from running separate regressions within each sample. However, because our standard errors account for autocorrelation of the residuals, the joint regression will have slightly different standard errors (for example, the joint regression will account for the fact that residuals are correlated from before and after Greece joined the ERM). The joint regression is preferable to separate regressions, because it uses more data and thus gives more precise standard errors.

price level as our dependent variable (which helps to obtain meaningful units).[25]

Table 2 in Appendix E describes the summary statistics by Euro/ERM status for the variables that enter our regression analysis. The Euro/ERM sample has 821 country-quarters and the non-Euro/ERM sample has 1120 country-quarters.[26] Both samples are unbalanced because a few countries have imperfect data coverage in earlier years (and because a few countries transition between samples). The two samples are comparable except that the non-Euro/ERM sample experienced slightly faster growth over the sample period.

House price shocks and demand recessions. Figure 6 plots the estimated sequences of coefficients by Euro/ERM status (see Figure 10 in Appendix E for the differenced coefficients). The panels at the top two rows illustrate our main empirical findings. The top left panel shows that, in the non-Euro/ERM sample (dashed blue line), a decline in house prices is followed by a sizeable and persistent decline in the policy interest rate. By contrast, in the Euro/ERM sample (solid red line), a decline in house prices does not lead to an additional decline in the country's interest rate relative to other Euro/ERM countries, illustrating the interest rate constraint.[27] The remaining panels in the top two rows illustrate that the shock is followed by a more severe demand recession in an Euro/ERM country than in a non-Euro/ERM country. In fact, the panels on GDP, investment, and consumption suggest that the shock initially leads to similar effects in both samples but is eventually followed by milder outcomes in the non-Euro/ERM sample.

These results are consistent with our prediction that risk premium shocks lead to a more severe demand recession when the interest rate is constrained. From the lens of our model, the interest rate policy mitigates a demand recession driven by a local risk premium shock outside the Eurozone but not within the Eurozone.[28]

[25] Earnings sometimes take a negative value (e.g., for Greece in recent years) which makes a log transformation problematic. Instead, we change the specification in (54) slightly so that the dependent variable is $(\text{earnings}_{t+h} - \text{earnings}_{t-1})/(\text{stock price}_{t-1})$. Likewise, we adjust the control variables that feature earnings by dividing them with the stock price at quarter $t-1$.

[26] These are the sample sizes for our baseline regression in which the outcome variable is the policy interest rate and the horizon is 0 (see (54)). For some regressions, the sample size is slightly smaller, because we estimate outcomes at future horizons (that removes some data from the end of the sample period) and because some variables do not have complete coverage.

[27] For the Euro era, the Euro/ERM-wide policy interest rate response is common to all countries and is captured by our time-fixed effects. And during the ERM era, there were severe cross-country monetary policy constraints. Figure 12 in Appendix E illustrates the results from the same regression without time-fixed effects. The figure shows that a negative house price shock in the Euro/ERM sample leads to a decline in the Euro/ERM-wide policy interest rate, but the magnitude of this decline is smaller than in the other sample. This is because house price shocks have a national (or idiosyncratic) component, and the Euro/ERM-wide policy interest rate arguably responds only to the Euro/ERM-wide (or systematic) component of these shocks.

[28] In our model, risk premium shocks generate a less severe recession in unconstrained countries because the interest rate policy response leads to a smaller decline in asset prices. This suggests that asset price changes might provide an inaccurate measure of the underlying shock. We believe our analysis is robust to this concern for three reasons. First, to the extent that this concern is relevant, it biases the empirical analysis against finding support for our mechanisms, because it implies that an equivalent magnitude of asset price decline corresponds to a larger underlying shock if the country has unconstrained interest rate. Second, the concern is less relevant in practice than in our model because the interest rate policy affects all assets, which implies that risk-driven price declines in one asset class (such as housing) are partially absorbed by price increases in other asset classes. Third, the concern is also less relevant for house prices because they seem to react to interest rate changes with some delay (see Figure 15 in Appendix E). In fact, the panel

Spillover effects and amplification. The panels at the bottom row of Figure 6 illustrate the effect of the house price shock on asset markets. The panels on earnings and stock prices establish that there are spillover effects to the stock market: specifically, earnings as well as stock prices decline more in the Euro/ERM sample than in the other sample (although the estimates are imprecise due to the high volatility of earnings and prices). The remaining panel illustrates that, after the initial shock, house prices decline more persistently and by a greater magnitude in the Euro/ERM sample.

These results are consistent with our prediction that the demand recession reduces firms' earnings and leads to a further decline in asset prices. From the lens of the model, stock prices (resp. house prices) decline less in the non-Euro/ERM sample due to the interest rate response, which not only increases the price to earnings ratio (resp. price to rent ratio) but also mitigates the recession and supports earnings (resp. rents).[29]

Speculation and further amplification. We need a proxy for speculation to test the final prediction of our model. We choose a measure of bank credit, which is a major catalyst of speculation in housing markets. First, banks can be thought of as the high-valuation investors ("optimists"), because they have a greater capacity and expertise to handle risk relative to non-institutional investors, and they have real estate exposures through mortgage loans. Under this interpretation, bank credit provides a measure of banks' exposure to the housing market. Second, banks also lend to other high-valuation investors in the housing markets such as optimistic homebuyers that use bank credit to purchase larger homes or second homes. When bank credit is easily available, perhaps because of banks' optimism about house prices, these high-valuation investors wield a greater influence in the housing market (see Simsek (2013a) for a formalization). Thus, bank credit provides a broad proxy for speculation in the housing market.

Our specific measure of bank credit comes from Baron and Xiong (2017), who construct a variable, "credit expansion", defined as the change in the bank credit to GDP ratio in the last three years. They standardize the variable by its mean and standard deviation within each country so that the measure is high when bank credit expansion in a country has been high in recent years relative to its historical trends. They show that their standardized measure predicts the likelihood of a large decline in bank equity prices, and despite the elevated risk, it also predicts lower average returns on bank equity. Their preferred interpretation is that bank equity investors are excessively optimistic or neglect crash risk, which in our framework would translate into greater speculation (by banks or their borrowers).

We use the BIS data on bank credit to households and nonfinancial firms to construct a close analogue of Baron and Xiong's standardized credit expansion variable (see Appendix E for details). We then run the same regressions as in (54), but we also include the interaction of the price shock

of Figure 6 on house prices suggests that the interest rate response only partially stabilizes risk-driven house price changes and with some delay.

[29] We cannot test the predictions on rents because we do not have reliable data.

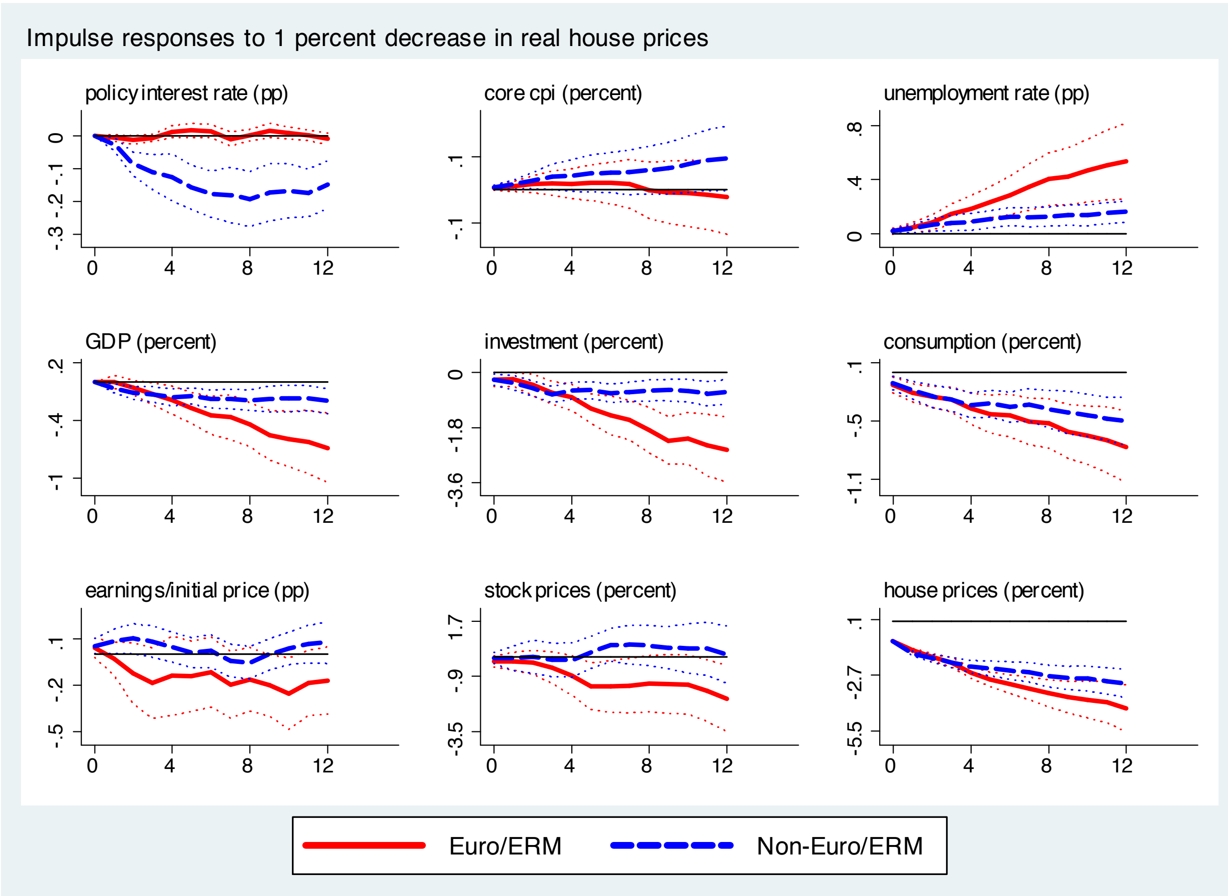

Figure 6: Results from the regression specification in (54) with the addition of the indicator variables for Euro/ERM and non-Euro/ERM status as well as the interaction of all right-hand-side variables with these indicators. The solid red (resp. dashed blue) lines plot the coefficients corresponding to the the negative log house price variable when the Euro/ERM status is equal to 1 (resp. 0). For the units, "percent" corresponds to 0.01 log units (i.e., it is approximate) and "pp" corresponds to percentage points. All regressions include time and country fixed effects; contemporaneous value and 12 lags of the first difference of log GDP; contemporaneous value and 12 lags of the policy interest rate; 12 lags of the first difference of log house prices; 12 lags of the first difference of the outcome variable. The dotted lines show 95% confidence intervals calculated according to Newey-West standard errors with a bandwidth of 20 quarters. All regressions are weighted by countries' PPP-adjusted GDP in 1990. Data is unbalanced quarterly panel that spans 1990Q1-2017Q4. All variables except for those in the top panel are adjusted for inflation. Earnings are normalized by the stock price at the quarter before the shock (see Footnote 25). The sources and the definitions of variables are described in Appendix E.

with standardized credit expansion. That is, we estimate,

$$Y_{j,t+h}^{h} - Y_{j,t-1}^{h} = \alpha_j^h + \gamma_t^h + \left[\begin{array}{c} \beta^{p,h} \left(-\Delta \log P_{j,t}\right) \\ +\beta^{pc,h} \left(-\Delta \log P_{j,t}\right) \times \text{credit expansion-std} \end{array} \right] + \beta^{c,h} \text{controls}_{j,t} + \varepsilon_{j,t}. \tag{55}$$

In addition to the earlier controls, we include 12 lags of standardized credit expansion to capture its direct impact. As before, we also interact all right-hand-side variables with the Euro/ERM and the non-Euro/ERM status indicators. We let $\beta_{euro}^{pc,h}$ and $\beta_{non}^{pc,h}$ denote the coefficient on the interaction of the shock and credit with these indicators. The sequence of coefficients, $\left\{\beta_{euro}^{pc,h}\right\}_{h=0}^{12}$ and $\left\{\beta_{non}^{pc,h}\right\}_{h=0}^{12}$, provide an estimate of the *additional* effect of the shock when credit expansion has been one standard deviation above average (relative to its baseline effect with average credit).

Figure 7 plots these sequences and illustrates our findings (see Figure 11 in the appendix for the differenced coefficients). The panels on the first two rows show that, in the Euro/ERM sample, house price shocks lead to a greater decline in economic activity when credit expansion has been high in recent years. In contrast, credit expansion does not seem to change the effect of the house price shock in the non-Euro/ERM sample. These results support our prediction that risk premium shocks lead to a more severe demand recession (in constrained economies) when they take place in an environment with elevated speculation.

On the other hand, the panels at the bottom row of Figure 7 present largely inconclusive results that do not necessarily support (or refute) our predictions. We do not find meaningful differences for the additional effect of house price shocks on earnings or house prices when credit expansion has been high (in either sample). We do find a negative effect on stock prices for the Euro/ERM sample, but the effect is not statistically significantly different from the other sample. That said, since standard errors are large, we cannot rule out sizeable effects either. Hence, while we tentatively conclude that speculation proxied by credit expansion is associated with deeper risk-centric demand recessions, further empirical research should verify the robustness of this conclusion as well as the precise channels by which speculation affects the recession.

Other supporting evidence. Our empirical analysis is related to Mian and Sufi (2014, 2018) who use regional data within the U.S. to provide evidence for the central role played by the house price cycle and housing speculation in the Great Recession.

Mian and Sufi (2014) argue that house price declines explain much of the job losses between 2007 and 2009. Our results for the Euro/ERM sample suggest that similar results hold in cross-country data, while the non-Euro/ERM sample suggests that monetary policy can mitigate the adverse effects of house price shocks. Moreover, while Mian and Sufi (2014) emphasize household deleveraging as the key channel by which house price declines cause damage, some of our empirical results (e.g., the investment response) suggest there are other mechanisms as well. As our model demonstrates, house price declines could lower aggregate demand even without household deleveraging or other financial frictions—although these additional ingredients would naturally amplify the effects.

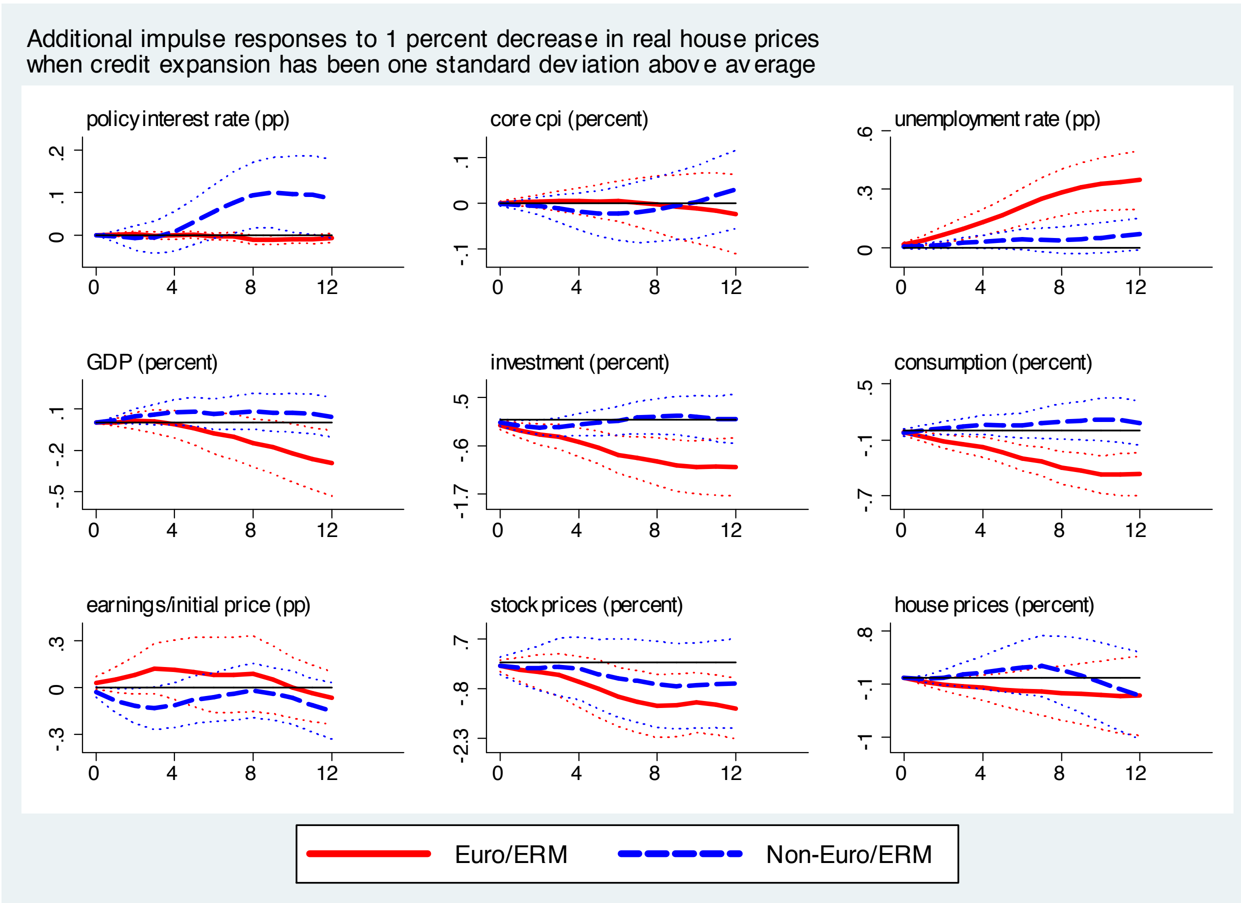

Figure 7: Results from the regression specification in (55) with the addition of the indicator variables for Euro/ERM and non-Euro/ERM status as well as the interaction of all right-hand-side variables with these indicators. The solid red (resp. dashed blue) lines plot the coefficients corresponding to the interaction of the negative log house price and the standardized credit expansion variables when the Euro/ERM status is equal to 1 (resp. 0). For the units, "percent" corresponds to 0.01 log units (i.e., it is approximate) and "pp" corresponds to percentage points. All regressions include time and country fixed effects; contemporaneous value and 12 lags of the first difference of log GDP; contemporaneous value and 12 lags of the policy interest rate; 12 lags of the first difference of log house prices; 12 lags of the first difference of the outcome variable; and 12 lags of standardized credit expansion. The dotted lines show 95% confidence intervals calculated according to Newey-West standard errors with a bandwidth of 20 quarters. All regressions are weighted by countries' PPP-adjusted GDP in 1990. Data is unbalanced quarterly panel that spans 1990Q1-2017Q4. All variables except for those in the top panel are adjusted for inflation. Earnings are normalized by the stock price at the quarter before the shock (see Footnote 25). The sources and the definitions of variables are described in Appendix E.

Mian and Sufi (2018) argue that housing speculation amplified the house price cycle and lead to a more severe downturn. As in our empirical exercise, they emphasize bank credit expansion as a major catalyst of speculation. They find that the U.S. areas more exposed to credit expansion in early 2000s featured greater speculative trading activity (measured from detailed transaction data) and greater belief disagreements (measured from survey data). They go on to argue that the same areas experienced a greater housing boom but also a much greater bust so they ended the housing cycle with lower house prices and economic activity. Our empirical results on speculation (although much less detailed) suggest similar results hold in cross-country data. Our model illustrates how greater speculation during the boom naturally leads to lower prices and economic activity once the economy transitions to recession.

In recent work, Pflueger et al. (2018) present evidence that suggests risk premium shocks in the *stock market* also affect aggregate demand and interest rates. Specifically, they construct a measure of risk appetite for the U.S. as the price of high (idiosyncratic) volatility stocks relative to low volatility stocks. They show that a decrease in their measure of risk appetite is followed by a slowdown in economic activity as well as a decline in the risk-free rate—similar to our results for the non-Euro/ERM sample. Pflueger et al. (2018) go on to argue that their risk appetite measure explains almost half of the variation of the one year risk-free rate in the U.S. since 1970. This suggests that the time varying risk premium is a quantitatively important driver of the risk-free rate in practice. Chodorow-Reich et al. (2019) provide further support for the link between the stock market and aggregate demand. Using regional data within the U.S., they find that a decline in local stock wealth (driven by aggregate stock prices) decreases local payroll and employment. They also find stronger effects in nontradable industries but no effects for tradable industries, consistent with a consumption wealth effect as in our model.

Focusing on a value-weighted average of house and stock prices, Jordà et al. (2019) argue that low frequency fluctuations in the risk premium in developed economies have been associated with a collapse of safe asset returns (as opposed to a spike in risky asset returns). In particular, when the risk premium rises, the risk-free rate tends to fall and the value-weighted average risky asset returns remain relatively stable, as in our model. Looking at more recent years, Del Negro et al. (2017) provide a comprehensive empirical evaluation of the different mechanisms that have put downward pressure on interest rates and show that risk and liquidity considerations played a central role (see also Caballero et al. (2017a)).

Finally, our mechanisms are supported by a literature that investigates the macroeconomic impact of "uncertainty shocks." Using vector autoregressions (VARs), Bloom (2009) shows that an increase in the volatility index in the U.S. is followed by a slowdown in economic activity. Moreover, although his model does not emphasize monetary policy, his empirical analysis shows that the shock is followed by a decline in the federal funds rate. This response suggests the effects could be more severe if the interest rate were constrained. Recent empirical work verifies this intuition and shows that uncertainty shocks in the U.S. are associated with a greater decline in economic activity when the federal funds rate is close to zero, arguably because of the zero lower bound constraint on the interest rate (see, for instance, Caggiano et al. (2017); Plante et al. (2018)).

8. Final remarks

We develop a risk-centric macroeconomic model to focus on the role of the *aggregate demand channel* in causing recessions driven by risky asset price fluctuations, and to study the effect of *financial speculation* on the severity of these recessions. In our model, when the interest rate is constrained, a rise in the risk premium lowers asset prices and triggers a demand recession, which further drives down asset prices. The feedbacks are especially powerful when investors are pessimistic and think the higher risk premium will persist. Hence, beliefs play a central role in the recession phase not only because they affect asset valuations but also because they determine the strength of the amplification mechanism. In the ex-ante boom phase, belief disagreements (and more broadly, heterogeneous valuations) matter because they induce investors to speculate. This speculation exacerbates the recession because it depletes high-valuation investors' wealth once the risk premium rises, which leads to a greater decline in asset prices and economic activity. Macroprudential policy (in the boom) improves outcomes by restricting speculation and preserving high-valuation investors' wealth during the recession. This policy intervention leads to a Pareto improvement because it internalizes the aggregate demand externalities that result from speculation.

Interest rate cuts in our model improve the market's Sharpe ratio. From this perspective, any policy that reduces perceived market volatility and prevents sudden asset price drops should have similar effect, providing support for various policies implemented during the aftermath of the subprime and European crises.

In our model, we use a lower bound constraint as the interest rate friction, but as we stated earlier our mechanisms are also applicable if the interest rate is constrained for other reasons. Also, when the interest rate has an upper bound as well as a lower bound (such as in a currency union or fixed exchange rate regime), our results often become stronger. In this setting, speculation creates damage not only by lowering asset prices during the recession but also by raising asset prices during the boom, when the aggregate demand is stretched above its natural level, which typically exacerbates the inefficiency. Moreover, in this case macroprudential policy during the boom is beneficial not only because it preserves high-valuation investors' wealth for a future recession but also because it immediately contains the excessive rise in asset prices.

In the main text, we did not take a stand on whether optimists or pessimists are right about the transition probabilities. The core of our analysis does not depend on this. For example, we could think of optimists as rational agents and pessimists as Knightian agents (see, e.g., Caballero and Krishnamurthy (2008); Caballero and Simsek (2013)). Absent any direct mechanism to alleviate Knightian behavior during severe recessions, the key point that reducing optimists's risk taking during the boom leads to Pareto improvements survives this alternative motivation.

As we noted earlier, our modeling approach belongs to the literature spurred by Brunnermeier and Sannikov (2014), although our analysis does not feature financial frictions. However, if we were to introduce these realistic frictions in our setting, many of the themes in that literature would reemerge and be exacerbated by aggregate demand feedbacks. For instance, in an incomplete markets setting, optimists take leveraged positions on the market portfolio and induce endogenous volatility in asset prices. In this case, a sequence of negative diffusion shocks that make the economy deeply pessimistic can lead to extreme tail events.

Finally, while this is mostly an applied theory paper, we also surveyed some of the extensive empirical evidence supporting our analysis, and provided our own evidence by contrasting the local response to risk premium shocks (captured by surprise house price changes) of (constrained) Euro/ERM countries to that of (unconstrained) non-Euro/ERM countries. Our evidence suggests that risk premium shocks lead to more severe recessions when the interest rate is constrained, as in our model. The evidence also supports our model's prediction that recessions reduce firms' earnings and lead to a further reduction in asset prices. Finally, we found some evidence consistent with our prediction that recessions are more severe when the shock takes place in an environment with high speculation (as measured by the size of the bank credit expansion before the shock).

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Online Appendices: Not for Publication

A. Appendix: Omitted Derivations for the Two Period Model

This appendix presents the derivations and proofs omitted from the main text for the two period model that we analyze in Section 2. We start by the case analyzed in the main text. We then analyze the case in which EIS is different than one, as well as the case with belief disagreements. Throughout, recall that the market portfolio is the claim to all output at date 1. Combining Eqs. (1) and (2), the return on the market portfolio is also log normally distributed, that is,

$$r^m(z_1) = \log\left(\frac{Q_1}{z_1}\right) \sim N\left(g - \log Q - \frac{\sigma^2}{2}, \sigma^2\right). \tag{A.1}$$

A.1. Baseline two period model

For this case, most of the analysis is provided in the main text. Here, we formally state the investor's problem and derive the optimality conditions. The investor takes the returns as given and solves the following problem,

$$\begin{aligned} &\max_{c_0, a_0, \omega^m} \log c_0 + e^{-\rho} \log U_1 \\ &\text{where } U_1 = \left(E\left[c_1(z_1)^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\ \text{s.t.} \quad & c_0 + a_0 = y_0 + Q \\ \text{and} \quad & c_1(z_1) = a_0\left(\omega^m \exp(r^m(z_1)) + (1-\omega^m)\exp\left(r^f\right)\right). \end{aligned}$$

Here, $c_1(z_1)$ denotes total financial wealth, which equals consumption (since the economy ends at date 1). Note that the investor has Epstein-Zin preferences with EIS coefficient equal to one and the RRA coefficient equal to $\gamma > 0$. The case with $\gamma = 1$ is equivalent to time-separable log utility as in the dynamic model.

In view of the Epstein-Zin functional form, the investor's problem naturally splits into two steps. Conditional on savings, a_0, she solves a portfolio optimization problem, that is, $U_1 = R^{CE} a_0$, where

$$\begin{aligned} R^{CE} &= \max_{\omega^m}\left(E\left[\left(R^p(z_1)\right)^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\ \text{and } R^p(z_1) &= \left(\omega^m \exp(r^m(z_1)) + (1-\omega^m)\exp\left(r^f\right)\right). \end{aligned} \tag{A.2}$$

Here, we used the observation that the portfolio problem is linearly homogeneous. The variable, $R^p(z_1)$, denotes the realized portfolio return per dollar, and R^{CE} denotes the optimal certainty-equivalent portfolio return. In turn, the investor chooses asset holdings, a_0, that solve the intertemporal problem,

$$\max_{a_0} \log(y_0 + Q - a_0) + e^{-\rho}\log\left(R^{CE} a_0\right). \tag{A.3}$$

The first order condition for this problem implies Eq. (4) in the main text. That is, regardless of her certainty-equivalent portfolio return, the investor consumes and saves a constant fraction of her lifetime wealth.

It remains to characterize the optimal portfolio weight, ω^m, as well as the certainty-equivalent return, R^{CE}. Even though the return on the market portfolio is log-normally distributed (see Eq. (A.1)), the portfolio return, $R^p(z_1)$, is in general not log-normally distributed (since it is the sum of a log-normal variable and a constant). Following Campbell and Viceira (2002), we assume the investor solves an approximate version of the portfolio problem ($A.2$) in which the log portfolio return is also normally distributed. To state the problem, let $\pi^p \equiv \log E[R^p] - r^f$ and $(\sigma^p)^2 \equiv var(\log R^p)$ to denote respectively the risk premium and the variance of the market portfolio (measured in log returns). Then, the approximate portfolio return satisfies,

$$\begin{aligned} \pi^p &= \omega^m \pi^k \\ \text{where } \pi^k &\equiv \log\left(E\left[\exp\left(r^m(z_1)\right)\right]\right) - r^f = E\left[r^m(z_1)\right] - r^f + \frac{\sigma^2}{2}. \end{aligned} \tag{A.4}$$

Hence, the risk premium on the portfolio return depends linearly on the risk premium on the market portfolio (measured in log returns). We also have,

$$\sigma^p = \omega^m \sigma. \tag{A.5}$$

Thus, the volatility of the portfolio also depends linearly on the volatility of the market portfolio (measured in log returns). These identities hold exactly in continuous time. In the two period model, they hold approximately when the period time-length is small. Moreover, they become exact for the level the risk premium that ensures equilibrium, $\omega^m = 1$, since in this case the portfolio return is actually log-normally distributed.

Taking the log of the objective function in problem ($A.2$), and using the log-normality assumption, the problem can be equivalently rewritten as,

$$\log R^{CE} - r^f = \max_{\omega^m} \pi^p - \frac{1}{2}\gamma\left(\sigma^p\right)^2, \tag{A.6}$$

where π^p and σ^p are defined in Eqs. ($A.4$) and ($A.5$). It follows that, up to an approximation (that becomes exact in equilibrium), the investor's problem turns into standard mean-variance optimization. Taking the first order condition, we obtain Eq. (6) in the main text. Substituting $\omega^m = 1$ and $E[r^m(z_1)] = g - \log Q - \frac{\sigma^2}{2}$ [cf. Eq. ($A.1$)] into this expression, we further obtain Eq. (7) in the main text. Substituting these expressions into ($A.6$), we also obtain the closed form solution for the certainty-equivalent return in (11).

A.2. More general EIS

In this case, the representative investor solves the following problem,

$$\begin{aligned} \max_{c_0, a_0, \omega^m, \{c_1(z_1)\}} \quad & U_0 = \frac{c_0^{1-1/\varepsilon} - 1}{1 - 1/\varepsilon} + e^{-\rho}\frac{U_1^{1-1/\varepsilon} - 1}{1 - 1/\varepsilon} \\ \text{where} \quad & U_1 = \left(E\left[c_1(z_1)^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\ \text{s.t.} \quad & c_0 + a_0 = y_0 + Q \\ \text{and} \quad & c_1(z_1) = a_0\left(\omega^m \exp\left(r^m(z_1)\right) + (1 - \omega^m)\exp\left(r^f\right)\right). \end{aligned}$$

Here, ε denotes the elasticity of substitution. The case with $\varepsilon = 1$ is equivalent to the earlier problem.

Most of the analysis remains unchanged. As before, the investor's problem splits into two parts. The portfolio problem ($A.2$) as well as its solution remains unchanged. In particular, Eqs. (6), (7), (11) from the main text continue to apply.

The main difference concerns the intertemporal problem ($A.3$), which is now given by,

$$\max_{a_0} (y_0 + Q - a_0)^{1-1/\varepsilon} + e^{-\rho} \left(R^{CE} a_0\right)^{1-1/\varepsilon}.$$

Taking the first order condition and rearranging terms, we obtain the consumption function,

$$c_0 = \frac{1}{1 + e^{-\rho\varepsilon} \left(R^{CE}\right)^{(\varepsilon-1)}} (y_0 + Q).$$

Combining this expression with the aggregate resource constraint, $y_0 = c_0$, we obtain the output-asset price relation (10) in the main text. The main difference from the earlier analysis is that consumption (and savings) also depends on income and substitution effects, in addition to the wealth effect in the main text. When $\varepsilon > 1$, the substitution effect dominates and all else equal an increase in the certainty-equivalent return reduces consumption (increases savings). This in turn lowers aggregate demand and output. Conversely, when $\varepsilon < 1$, the income effect dominates and an increase in certainty-equivalent return increases consumption, aggregate demand, and output.

The equilibrium is found by jointly solving Eq. (10) together with Eqs. (7) and (11), as well as the constrained policy interest rate. Collecting the equations together, the equilibrium tuple, $\left(y_0, Q, R^{CE}, r^f\right)$, is the solution to the following system,

$$\begin{aligned}
\log y_0 &= \rho\varepsilon + (1-\varepsilon)\log R^{CE} + \log Q \qquad (A.7)\\
\log R^{CE} &= g - \log Q - \frac{1}{2}\gamma\sigma^2\\
\sigma &= \frac{1}{\gamma}\frac{g - \log Q - r^f}{\sigma}\\
r^f &= \max\left(r^{f*}, 0\right) \text{ where } r^{f*} \text{ ensures } y_0 = z_0.
\end{aligned}$$

To characterize the solution further, consider the case in which the equilibrium is supply determined, $y_0 = z_0 = 1$. Substituting this into the first two equations, we solve for the first-best price level of the market portfolio as,

$$\log Q^* = -\rho + \frac{(\varepsilon - 1)}{\varepsilon}\left(g - \frac{1}{2}\gamma\sigma^2\right). \tag{A.8}$$

Substituting this into the last equation, we further obtain an expression for "rstar",

$$\begin{aligned}
r^{f*} &= \rho + g - \gamma\sigma^2 - \frac{(\varepsilon-1)}{\varepsilon}\left(g - \frac{1}{2}\gamma\sigma^2\right) \qquad (A.9)\\
&= \rho + \frac{g}{\varepsilon} - \frac{1}{2}\gamma\left(1 + \frac{1}{\varepsilon}\right)\sigma^2.
\end{aligned}$$

Note that setting $\varepsilon = 1$ recovers Eq. (8) in the main text. The main difference is that "rstar" is now also influenced by the attractiveness of investment opportunities, captured by the term $g - \frac{1}{2}\gamma\sigma^2$ (that shifts $\log R^{CE}$). When $\varepsilon > 1$, reducing the attractiveness of investment opportunities induces the representative household to consume more and save less due to a substitution effect. This requires an increase in the risk-free rate to equilibrate the goods market. In this case, a risk premium shock that increases γ or σ (or lowers g) reduces aggregate wealth, which tends to reduce the interest rate as before, but it also reduces the attractiveness of investment opportunities, which tends to raise the interest rate. When $\varepsilon < 1$, the two channels work in the same direction. The second line of Eq. ($A.9$) collects similar terms together and shows that the risk shocks lower "rstar" as in the baseline setting regardless of the level of ε. When $\varepsilon > 1$, the effect is quantitatively weaker due to the substitution channel but it is qualitatively the same.

Now consider the case in which the interest rate is at its lower bound, $r^f = 0$. Substituting this into the equation system (A.7), we obtain,

$$\begin{aligned} \log Q &= g - \gamma\sigma^2 \qquad\qquad (A.10) \\ \text{and } \log y_0 &= \varepsilon\left(\rho + \log Q - \frac{(\varepsilon-1)}{\varepsilon}\left(g - \frac{1}{2}\gamma\sigma^2\right)\right) \\ &= \varepsilon\left(\rho + \frac{g}{\varepsilon} - \frac{1}{2}\gamma\left(1 + \frac{1}{\varepsilon}\right)\sigma^2\right). \end{aligned}$$

In this case, the additional effect of the changes in the attractiveness of investment opportunities is absorbed by output, because the interest rate does not respond. An increase in γ or σ (or a decrease in g) tends to reduce the output by reducing the aggregate wealth, as in the baseline setting, but it also affects output through substitution or income effects. The last line in (A.10) illustrates that the wealth effect dominates regardless of the level of ε. When $\varepsilon > 1$, the substitution effect mitigates the quantitative impact of the wealth effect relative to the baseline setting but it does not overturn it. When $\varepsilon < 1$, the income effect amplifies the quantitative impact of the wealth effect.

A.3. Belief disagreements and speculation

We denote optimists and pessimists respectively with superscript $i \in \{o, p\}$. With a slight abuse of notation, we also let $\alpha^o \equiv \alpha$ and $\alpha^p \equiv 1 - \alpha$ denote respectively optimists' and pessimists' wealth shares. Recall that investors are identical except possibly their beliefs about aggregate growth. Then, type i investors solve the following problem,

$$\begin{aligned} \max_{c_0, a_0, \omega^m, \{c_1(z_1)\}} \quad & \log c_0 + e^{-\rho}\log U_1 \qquad\qquad (A.11) \\ \text{where} \quad & U_1 = \left(E^i\left[c_1(z_1)^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\ \text{s.t.} \quad & c_0 + a_0 = \alpha^i(y_0 + Q) \\ \text{and} \quad & c_1(z_1) = a_0\left(\omega^m \exp\left(r^m(z_1)\right) + (1-\omega^m)\exp\left(r^f\right)\right). \end{aligned}$$

Note that we set the EIS equal to one as in the baseline setting. Note also that the asset market clearing condition requires,

$$\omega^{m,o} a_0^o + \omega^{m,p} a_0^p = Q, \qquad (A.12)$$

that is, the total amount of wealth invested in the market portfolio equals the value of the market portfolio. The rest of the model is the same as in the baseline setting.

In this case, the investor's portfolio problem (A.2) remains unchanged. Applying the log-normal approximation that we described previously, we obtain Eq. (6) as in the main text, that is,

$$\omega^{m,i}\sigma \simeq \frac{1}{\gamma}\frac{E^i\left[r^m(z_1)\right] + \frac{\sigma^2}{2} - r^f}{\sigma}.$$

Substituting $E^i[r^m(z_1)] = g^i - \log Q - \frac{\sigma^2}{2}$ [cf. Eq. ($A.1$)] into this expression, we further obtain,

$$\omega^{m,i}\sigma \simeq \frac{1}{\gamma}\frac{g^i - \log Q - r^f}{\sigma}. \tag{A.13}$$

As before, investors choose their share of the market portfolio so that their optimal portfolio risk is proportional to the Sharpe ratio. The difference is that the Sharpe ratio is calculated according to investors' own beliefs (and it is greater for optimists since $g^o > g^p$).

The intertemporal problem ($A.3$) also remains unchanged. Taking the first order condition, we obtain,

$$c_0^i = \frac{1}{1+e^{-\rho}}\alpha^i(y_0 + Q) \tag{A.14}$$

Aggregating this equation across investors, and using the aggregate resource constraint (3), shows that the output-asset price relation (5) continues to apply in this setting. Belief heterogeneity does not affect this equation since investors share the same discount rate, ρ.

Next note that combining ($A.12$), ($A.14$) and (5), the asset market clearing condition can be rewritten as,

$$\alpha\omega^{m,o} + (1-\alpha)\omega^{m,p} = 1. \tag{A.15}$$

Investors' wealth-weighted average portfolio weight on the market portfolio is equal to one. Combining this with Eq. ($A.13$), we obtain the following analogue of Eq. (7),

$$\sigma \simeq \frac{1}{\gamma}\frac{\alpha^o g^o + \alpha^p g^p - \log Q - r^f}{\sigma}. \tag{A.16}$$

Hence, the risk balance condition continues to apply with the difference that the expected growth rate is determined according to a weighted average belief. Another difference is that the condition is typically not exact because investors' shares of the market portfolio typically deviate from one (and thus, their return is typically not log-normal). Specifically, the equilibrium portfolio shares satisfy, $\omega^o > 1 > \omega^p$: optimists' make a leveraged investment in the market portfolio by issuing some risk-free debt, whereas pessimists invest only a fraction of their wealth in the market portfolio (and invest the rest of their wealth in the risk-free asset issued by optimists).

Next consider the supply-determined equilibrium in which output is equal to its potential, $y_0 = z_0 = 1$. By Eq. (5), this requires the asset price to be at a particular level, $Q^* = e^{-\rho}$. Combining this with Eq. ($A.16$) we obtain Eq. (12) in the main text that characterizes "rstar." The level of "rstar" is increasing in optimists' wealth share, α. This is because increasing optimists' wealth share tends to increase asset prices, aggregate demand, and output. In a supply-determined equilibrium, the monetary policy increases the interest rate to neutralize the impact of optimists on aggregate demand and output.

Finally, consider the case in which the interest rate is at its lower bound, $r^f = 0$. Substituting this into the risk balance condition ($A.16$), and using the output-asset price relation (5), we obtain Eq. (9) in the main text that characterizes the equilibrium level of output in a demand recession. In this case, increasing optimists' wealth share translates into an actual increase in asset prices, aggregate demand, and output, because the monetary policy cannot neutralize these effects due to the constraint on the interest rate.

B. Appendix: Omitted Derivations for the Dynamic Model

This appendix presents the details of the dynamic model that we present and analyze in Sections 3-5. Sections B.1-B.3 describe derivations and proofs omitted from the main text for the dynamic model that. Section B.4 describes how we parameterize the model. The subsequent appendix C presents the details of the welfare analysis for the same model.

B.1. Omitted derivations in Section 3

B.1.1. Portfolio problem and its recursive formulation

The investor's portfolio problem (at some time t and state s) can be written as,

$$V_{t,s}^{i}\left(a_{t,s}^{i}\right)=\max_{\left[\tilde{c}_{\tilde{t},\tilde{s}},\tilde{\omega}_{\tilde{t},\tilde{s}}^{m},\tilde{\omega}_{\tilde{t},\tilde{s}}^{\tilde{s}'}\right]_{\tilde{t}\geq t,\tilde{s}}} E_{t,s}^{i}\left[\int_{t}^{\infty}e^{-\rho\tilde{t}}\log\tilde{c}_{\tilde{t},\tilde{s}}^{i}d\tilde{t}\right]$$

$$\text{s.t.}\begin{cases} da_{t,s}^{i}=\left(a_{t,s}^{i}\left(r_{t,s}^{f}+\tilde{\omega}_{t,s}^{m}\left(r_{t,s}^{m}-r_{t,s}^{f}\right)-\tilde{\omega}_{\tilde{t},\tilde{s}}^{\tilde{s}'}\right)-\tilde{c}_{t,s}\right)dt+\tilde{\omega}_{t,s}^{m}a_{t,s}^{i}\sigma_{s}dZ_{t} & \text{absent transition,}\\ a_{t,s'}^{i}=a_{t,s}^{i}\left(1+\tilde{\omega}_{t,s}^{m}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}+\tilde{\omega}_{t,s}^{s'}\frac{1}{p_{t,s}^{s'}}\right) & \text{if there is a transition to state } s'\neq s.\end{cases} \tag{B.1}$$

Here, $E_{t,s}^{i}\left[\cdot\right]$ denotes the expectations operator that corresponds to the investor i's beliefs for state transition probabilities. The HJB equation corresponding to this problem is given by,

$$\begin{aligned}\rho V_{t,s}^{i}\left(a_{t,s}^{i}\right)=&\max_{\tilde{\omega}^{m},\tilde{\omega}^{s'},\tilde{c}}\log\tilde{c}+\frac{\partial V_{t,s}^{i}}{\partial a}\left(a_{t,s}^{i}\left(r_{t,s}^{f}+\tilde{\omega}^{m}\left(r_{t,s}^{m}-r_{t,s}^{f}\right)-\tilde{\omega}^{s'}\right)-\tilde{c}\right)\\&+\frac{1}{2}\frac{\partial^{2}V_{t,s}^{i}}{\partial a^{2}}\left(\tilde{\omega}^{m}a_{t,s}^{i}\sigma_{s}\right)^{2}+\frac{\partial V_{t,s}^{i}\left(a_{t,s}^{i}\right)}{\partial t}\\&+\lambda_{s}^{i}\left(V_{t,s'}^{i}\left(a_{t,s}^{i}\left(1+\tilde{\omega}^{m}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}+\frac{\tilde{\omega}^{s'}}{p_{t,s}^{s'}}\right)\right)-V_{t,s}^{i}\left(a_{t,s}^{i}\right)\right).\end{aligned} \tag{B.2}$$

In view of the log utility, the solution has the functional form in (45), which we reproduce here,

$$V_{t,s}^{i}\left(a_{t,s}^{i}\right)=\frac{\log\left(a_{t,s}^{i}/Q_{t,s}\right)}{\rho}+v_{t,s}^{i}.$$

The first term in the value function captures the effect of holding a greater capital stock (or greater wealth), which scales the investor's consumption proportionally at all times and states. The second term, $v_{t,s}^{i}$, is the normalized value function when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^{i}=Q_{t,s}$). This functional form also implies,

$$\frac{\partial V_{t,s}^{i}}{\partial a}=\frac{1}{\rho a_{t,s}^{i}}\text{ and }\frac{\partial^{2}V_{t,s}^{i}}{\partial a^{2}}=\frac{-1}{\rho\left(a_{t,s}^{i}\right)^{2}}.$$

The first order condition for $\tilde{c}$ then implies Eq. (22) in the main text. The first order condition for $\tilde{\omega}^{m}$ implies,

$$\frac{\partial V_{t,s}^{i}}{\partial a}a_{t,s}^{i}\left(r_{t,s}^{m}-r_{t,s}^{f}\right)+\lambda_{s}^{i}\frac{\partial V_{t,s'}^{i}\left(a_{t,s'}^{i}\right)}{\partial a}a_{t,s}^{i}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}=-\frac{\partial^{2}V_{t,s}^{i}}{\partial a^{2}}\omega_{t,s}^{m}\left(a_{t,s}^{i}\sigma_{s}\right)^{2}.$$

After substituting for $\frac{\partial V_{t,s}^{i}}{\partial a}$, $\frac{\partial V_{t,s'}^{i}}{\partial a}$, $\frac{\partial^{2}V_{t,s}^{i}}{\partial a^{2}}$ and rearranging terms, this also implies Eq. (23) in the main text.

Finally, the first order condition for $\tilde{\omega}^{s'}$ implies,

$$\frac{p_{t,s}^{s'}}{\lambda_s^i} = \frac{\frac{\partial V_{t,s'}^i\left(a_{t,s'}^i\right)}{\partial a}}{\frac{\partial V_{t,s}^i\left(a_{t,s}^i\right)}{\partial a}} = \frac{1/a_{t,s'}^i}{1/a_{t,s}^i},$$

which is Eq. (24) in the main text. This completes the characterization of the optimality conditions.

B.1.2. New Keynesian microfoundation for nominal rigidities

The supply side of our model features nominal rigidities similar to the standard New Keynesian setting. There is a continuum of measure one of monopolistically competitive production firms denoted by ν. These firms own the capital stock (in equal proportion) and produce differentiated goods, $y_{t,s}(\nu)$, subject to the technology,

$$y_{t,s}(\nu) = A\eta_{t,s}(\nu)\, k_{t,s}. \tag{B.3}$$

Here, $\eta_{t,s}(\nu) \in [0,1]$ denotes the firm's choice of capital utilization. We assume utilization is free up to $\eta_{t,s}(\nu) = 1$ and infinitely costly afterwards. The production firms sell their output to a competitive sector that produces the final output according to the CES technology,

$$y_{t,s} = \left(\int_0^1 y_{t,s}(\nu)^{\frac{\varepsilon-1}{\varepsilon}}\, d\nu\right)^{\varepsilon/(\varepsilon-1)}, \tag{B.4}$$

for some $\varepsilon > 1$. Thus, the demand for the firms' goods implies,

$$y_{t,s}(\nu) \leq p_{t,s}(\nu)^{-\varepsilon}\, y_{t,s}, \text{ where } p_{t,s}(\nu) = P_{t,s}(\nu)/P. \tag{B.5}$$

Here, $p_{t,s}(\nu)$ denotes the firm's relative price, which depends on its nominal price, $P_{t,s}(\nu)$, as well as the ideal nominal price index, $P_{t,s} = \left(\int P_{t,s}(\nu)^{1-\varepsilon}\, d\nu\right)^{1/(1-\varepsilon)}$. We write the demand constraint as an inequality because an individual firm can in principle refuse to meet the demand for its goods.

Without price rigidities, the firm chooses $\eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu), p_{t,s}(\nu)$ to maximize its earnings, $p_{t,s}(\nu)\, y_{t,s}(\nu)$, subject to the supply constraint in ($B.3$) and the demand constraint, ($B.5$). In this case, the demand constraint holds as equality (because otherwise the firm can always raise its price to keep its production unchanged and raise its earnings). By combining the constraints, the firm's problem can be written as,

$$\max_{p_{t,s}(\nu),\eta_{t,s}(\nu)} p_{t,s}(\nu)^{1-\varepsilon}\, y_{t,s} \text{ s.t. } 0 \leq \eta_{t,s}(\nu) = \frac{p_{t,s}(\nu)^{-\varepsilon}\, y_{t,s}}{Ak_{t,s}} \leq 1.$$

Inspecting this problem reveals that the solution features full factor utilization, $\eta_{t,s}(\nu) = 1$. This is because, when $\eta_{t,s}(\nu) < 1$, the marginal cost of production is zero. Thus, the firm can always lower its price and increase its demand and production, which in turn increases its earnings. Hence, at the optimum, the firms set $\eta_{t,s}(\nu) = 1$ and $y_{t,s}(\nu) = Ak_{t,s}$. To produce at this level, they set the relative price level, $p_{t,s}(\nu) = \left(\frac{y_{t,s}}{Ak_{t,s}}\right)^{-1/\varepsilon}$. Since all firms are identical, we also have $p_{t,s}(\nu) = 1$ and $y_{t,s} = y_{t,s}(\nu) = Ak_{t,s}$. In particular, output is determined by aggregate supply at full factor utilization.

Now consider the alternative setting in which firms have a preset nominal price that is equal for all firms, $P_{t,s}(\nu) = P$. This also implies the relative price of a firm is fixed and equal to one, $p_{t,s}(\nu) = 1$. The firm chooses the remaining variables, $\eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu)$, to maximize its earnings, $y_{t,s}(\nu)$, subject to the supply constraint in ($B.3$) and the demand constraint, ($B.5$). Combining the constraints and using

$p_{t,s}(\nu) = 1$, the firm's problem can be written as,

$$\max_{\eta_{t,s}(\nu)} A\eta_{t,s}(\nu) k_{t,s} \text{ s.t. } 0 \leq \eta_{t,s}(\nu) \leq 1 \text{ and } A\eta_{t,s}(\nu) k_{t,s} \leq y_{t,s}.$$

The solution is given by, $\eta_{t,s}(\nu) = \min\left(1, \frac{y_{t,s}}{Ak_{t,s}}\right)$. Intuitively, when $\eta_{t,s}(\nu) < 1$ and $A\eta_{t,s}(\nu) k_{t,s} < y_{t,s}$, the marginal cost of production is zero and there is some unmet demand for firms' goods. The firm optimally increases its production until the supply or the demand constraints bind. Combining this observation with the production technology for the final output, we also obtain, $y_{t,s} \leq Ak_{t,s}$. This implies that the demand constraint holds as equality also in this case. In particular, we have $\eta_{t,s}(\nu) = \frac{y_{t,s}}{Ak_{t,s}} \leq 1$.

In sum, when the firms' nominal prices are fixed, aggregate output is determined by aggregate demand subject to the capacity constraint, which verifies Eq. (19) in the dynamic model (and Eq. (3) in the two period model).

Note also that, in equilibrium, firms' equilibrium earnings are equal to aggregate output, $y_{t,s}$. Since firms own the capital (and there is no rental market for capital), the division of these earnings between return to capital and monopoly profits is indeterminate. This division does not play an important role in our baseline model but it matters when we introduce investment. In Appendix D with endogenous investment (that we present subsequently), we use slightly different microfoundations that ensure earnings accrue to firms in the form of return to capital, i.e., there are no monopoly profits, which helps to simplify the exposition.

B.2. Omitted derivations in Section 4

Proof of Proposition 1. Provided in the main text.

B.3. Omitted derivations in Section 5

We derive the equilibrium conditions that we state and use in Section 5. First note that, using Eq. (24), the optimality condition (23) can be written as,

$$\omega^{m,i}_{t,s}\sigma_s = \frac{1}{\sigma_s}\left(r^m_{t,s} - r^f_{t,s} + p^{s'}_{t,s}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right). \tag{B.6}$$

Note also that Eq. (25) implies,

$$\omega^{m,o}_{t,s} = \omega^{m,p}_{t,s} = 1. \tag{B.7}$$

Next note that by definition, we have

$$a^o_{t,s} = \alpha_{t,s}Q_{t,s}k_{t,s} \text{ and } a^p_{t,s} = (1-\alpha_{t,s})\,Q_{t,s}k_{t,s} \text{ for each } s \in \{1,2\}.$$

After plugging these into Eq. (24), using $k_{t,s} = k_{t,s'}$ (since capital does not jump), and aggregating over optimists and pessimists, we obtain,

$$p^{s'}_{t,s} = \overline{\lambda}_{t,s}\frac{Q_{t,s}}{Q_{t,s'}}, \tag{B.8}$$

where $\overline{\lambda}_{t,s}$ denotes the wealth-weighted average belief defined in (38).

Next, we combine Eqs. $(B.6)$, $(B.7)$, and $(B.8)$ to obtain

$$\sigma_s = \frac{1}{\sigma_s}\left(r^m_{t,s} - r^f_{t,s} + \overline{\lambda}_{t,s}\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right)\right) \text{ for each } s \in \{1,2\}. \tag{B.9}$$

Substituting for $r^m_{t,s}$ from Eq. (28), we obtain the risk balance condition (39) in the main text.

We next characterize investors' equilibrium positions. Combining Eq. ($B.1$) with Eqs. ($B.7$) and ($B.8$), investors' wealth after transition satisfies,

$$\frac{a^i_{t,s'}}{a^i_{t,s}} = \frac{Q_{t,s'}}{Q_{t,s}} \left(1 + \frac{\omega^{s',i}_{t,s}}{\overline{\lambda}_{t,s}}\right). \tag{B.10}$$

From Eq. (24), we have $\frac{p^{s'}_{t,s}}{\lambda^i_s} = \frac{1/a^i_{t,s'}}{1/a^i_{t,s}}$. Substituting this into the previous expression and using Eq. ($B.8$) once more, we obtain,

$$\omega^{s',i}_{t,s} = \lambda^i_s - \overline{\lambda}_{t,s} \text{ for each } i \in \{o, p\}. \tag{B.11}$$

Combining this with Eq. (38) implies Eq. (40) in the main text.

Finally, we characterize the dynamics of optimists' wealth share. Combining Eqs. ($B.10$) and ($B.11$) implies,

$$\frac{a^i_{t,s'}}{a^i_{t,s}} = \frac{\lambda^i_s}{\overline{\lambda}_{t,s}} \frac{Q_{t,s'}}{Q_{t,s}}. \tag{B.12}$$

Combining this with the definition of wealth shares as well as $k_{t,s} = k_{t,s'}$, we further obtain,

$$\frac{\alpha_{t,s'}}{\alpha_{t,s}} = \frac{\lambda^o_s}{\overline{\lambda}_{t,s}}. \tag{B.13}$$

Thus, it remains to characterize the dynamics of wealth conditional on no transition. To this end, we combine Eq. ($B.1$) with Eqs. ($B.7$) , (28) , (22) to obtain,

$$\frac{da^o_{t,s}}{a^o_{t,s}} = \left(g + \mu^Q_{t,s} - \omega^{s',i}_{t,s}\right) dt + \sigma_s dZ_t.$$

After substituting $a^o_{t,s} = \alpha_{t,s} Q_{t,s} k_{t,s}$, and using the observation that $\frac{dQ_{t,s}}{Q_{t,s}} = \mu^Q_{t,s} dt$ and $\frac{dk_{t,s}}{k_{t,s}} = gdt + \sigma_s dZ_t$, we further obtain,

$$\frac{d\alpha_{t,s}}{\alpha_{t,s}} = -\omega^{s',o}_{t,s} dt = -\left(\lambda^o_s - \overline{\lambda}_{t,s}\right) dt. \tag{B.14}$$

Combining Eqs. ($B.13$) and ($B.14$) implies Eq. (41) in the main text.

Proof of Proposition 2. First consider the high-risk-premium state, $s = 2$. Combining Eqs. (41) and (42), we obtain the differential equation system,

$$\begin{aligned} \dot{q}_{t,2} &= -\left(\rho + g + \overline{\lambda}_2 (\alpha_{t,2}) \left(1 - \frac{\exp(q_2)}{Q^*}\right) - \sigma_2^2\right), \\ \dot{\alpha}_{t,2} &= -\left(\lambda^o_2 - \lambda^p_2\right) \alpha_{t,2} (1 - \alpha_{t,2}). \end{aligned} \tag{B.15}$$

This system describes the joint dynamics of the price and optimists' wealth share, $(q_{t,2}, \alpha_{t,2})$, conditional on there not being a transition. We next analyze the solution to this system using the phase diagram over the range $\alpha \in [0, 1]$ and $q_2 \in [q^p_2, q^o_2]$. Here, recall that q^i_2 corresponds to the equilibrium log price with common beliefs characterized in Section 4 corresponding to type i investors' belief.

First note that the system has two steady states given by, $(\alpha_{t,2} = 0, q_{t,2} = q^p_2)$, and $(\alpha_{t,2} = 1, q_{t,2} = q^o_2)$. Next note that the system satisfies the Lipschitz condition over the relevant range. Thus, the vector flows that describe the law of motion do not cross. Next consider the locus, $\dot{q}_2 = 0$. By comparing Eqs. (42)

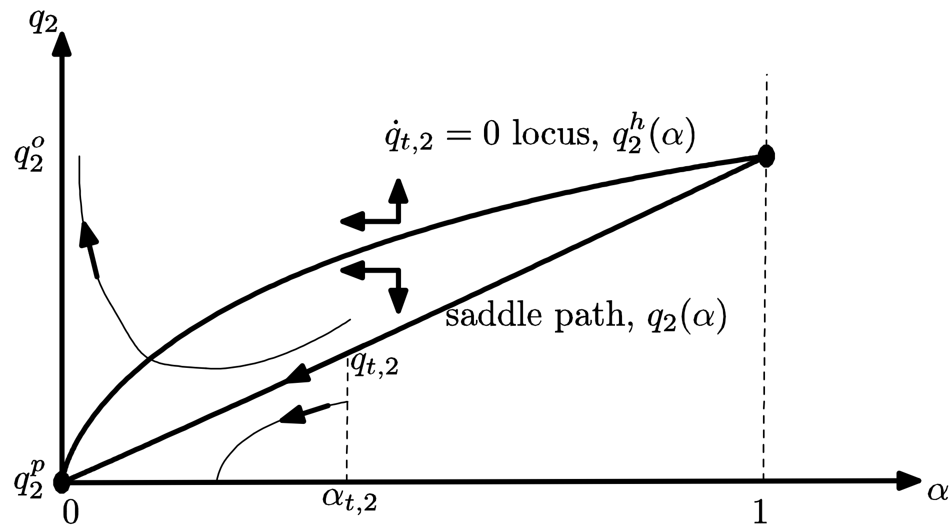

Figure 8: The phase diagram that describes the equilibrium with heterogeneous beliefs.

and (33), this locus is exactly the same as the price that would obtain if investors shared the same wealth-weighted average belief, denoted by $q_2 = q_2^h(\alpha)$. Using our analysis in Section 4, we also find that $q_2^h(\alpha)$ is strictly increasing in α. Moreover, $q_2 < q_2^h(\alpha)$ implies $\dot{q}_{t,2} < 0$ whereas $q_2 > q_2^h(\alpha)$ implies $\dot{q}_{t,2} > 0$. Finally, note that $\dot{\alpha}_{t,2} < 0$ for each $\alpha \in (0,1)$.

Combining these observations, the phase diagram has the shape in Figure 8. This in turn implies that the system is saddle path stable. Given any $\alpha_{t,2} \in [0,1)$, there exists a unique solution, $q_{t,2}$, which ensures that $\lim_{t\to\infty} q_{t,2} = q_2^p$. We define the price function (the saddle path) as $q_2(\alpha)$. Note that the price function satisfies $q_2(\alpha) < q_2^h(\alpha)$ for each $\alpha \in (0,1)$, since the saddle path cannot cross the locus, $\dot{q}_{t,2} = 0$. Note also that $q_2(1) = q_2^o$, since the saddle path crosses the other steady-state, $(\alpha_{t,2} = 1, q_{t,2} = q_2^o)$. Finally, recall that $q_2 < q_2^h(\alpha)$ implies $\dot{q}_{t,2} < 0$. Combining this with $\dot{\alpha}_{t,2} < 0$, we further obtain $\frac{dq_2(\alpha)}{d\alpha} > 0$ for each $\alpha \in (0,1)$.

Next note that, after substituting $\dot{q}_{t,2} = q_2'(\alpha)\dot{\alpha}_{t,2}$, Eq. ($B.15$) implies the differential equation (43) in α-domain. Thus, the above analysis shows there exists a solution to the differential equation with $q_2(0) = q_2^p$ and $q_2(1) = q_2^o$. Moreover, the solution is strictly increasing in α, and it satisfies $q_2(\alpha) < q_2^h(\alpha)$ for each $\alpha \in (0,1)$. Note also that this solution is unique since the saddle path is unique. The last part of the proposition follows from Eqs. (26) and (27).

Next consider the low-risk-premium state, $s = 1$. In the conjectured equilibrium, we have $Q_{t,1} = Q^*$, which also implies $\mu_{t,1}^Q = 0$. Substituting these expressions into Eq. (39), we obtain the risk balance condition in this state,

$$\sigma_1 = \frac{1}{\sigma_1}\left(g + \rho - r_{t,1}^f + \overline{\lambda}_{t,1}\left(1 - \frac{Q^*}{Q_{t,2}}\right)\right).$$

Writing the equilibrium variables as a function of optimists' wealth share, we obtain $r_{t,1}^f = r_1^f(\alpha)$ and $\overline{\lambda}_{t,1} = \overline{\lambda}_1(\alpha)$ and $Q_{t,2} = \exp(q_2(\alpha'))$, where $\alpha' = \alpha\lambda_1^o/\overline{\lambda}_1(\alpha)$ denotes optimists' wealth share after a transition [cf. Eq. (41)]. Substituting these expressions into the risk balance condition and rearranging terms, we obtain Eq. (44) in the main text that, which we replicate here,

$$r_1^f(\alpha) = \rho + g - \overline{\lambda}_1(\alpha)\left(\frac{Q^*}{\exp(q_2(\alpha'))} - 1\right) - \sigma_1^2.$$

Note also that $\frac{dr_1^f(\alpha)}{d\alpha} > 0$ because $\overline{\lambda}_1(\alpha)$ is decreasing in α (in view of Assumption 4), and $q_2(\alpha')$ is strictly increasing in α. The latter observation follows since $\alpha' = \frac{\alpha\lambda_1^o}{\alpha\lambda_1^o+(1-\alpha)\lambda_1^p}$ is increasing in α (in view of Assumption 4) and $q_2(\cdot)$ is a strictly increasing function. Note also that $r_1^f(\alpha) > r_1^f(0) > 0$, where the latter inequality follows since Assumptions 1-3 holds for the pessimistic belief. Thus, the interest rate in state 1 is always positive, which verifies our conjecture and completes the proof. □

B.4. Details of the parameterization

This section describes the details of the parameterization of the dynamic model that we use to numerically illustrate our findings. This parameterization is only meant to be reasonable, as its purpose is to give a sense of potential magnitudes. Throughout, we measure time in years so that the continuous-time rates we choose correspond to (approximate) yearly rates.

Since we do not explicitly model steady-state inflation (for simplicity), we interpret the growth rate in our model as $g = \tilde{g} + \pi$ where $\tilde{g}$ can be thought of as the real growth rate and $\pi \geq 0$ can be thought of as the steady-state inflation. With this adjustment, we can interpret the returns in our model as capturing the corresponding *nominal* returns in the data. In particular, the zero lower bound constraint in the model (20) becomes a restriction on the *nominal* risk-free rate (as in the data). We set $\pi = 2\%$ based on the Fed's inflation target in recent decades; and $\tilde{g} = 3\%$ based on pre-recession estimates for the U.S. trend output growth, which leads to:

$$g = 5\%.$$

For the discount rate, we set,

$$\rho = 4\%,$$

based on the yearly discount rates typically assumed in the literature. This implies a first-best (nominal) return to capital given by $r^{m,*} = \rho + r = 9\%$ [cf. (30)], which is consistent with the historical estimates for the weighted-average return on stocks and housing in Jordà et al. (2019).

We set the variance in the low-risk-premium state to target the first-best nominal risk-free interest rate in the boom given by, $r_1^{f,*} = 4\%$ (equivalently, a real risk-free rate given by 2%), which is consistent with the low interest rates in recent years. Using (31), this leads to:

$$\sigma_1^2 = 5\%.$$

We set the variance in the high-risk-premium state to target a first-best interest rate $r_2^{f,*} = -1\%$, which leads to:

$$\sigma_2^2 = 10\%.$$

These choices (together with the choices of ρ and g) ensure that Assumption 1 holds. For the productivity level, we set $A = 1$. This does not play a role as it scales all variables.

It remains to set investors' beliefs for transition probabilities, $\left(\lambda_s^i\right)_{s\in\{1,2\},i\in\{o,p\}}$. We set:

$$\begin{aligned}
\lambda_1^o &= 1/10 \text{ and } \lambda_2^o = 1/3, \\
\lambda_1^p &= 1/3 \text{ and } \lambda_2^p = 1/10.
\end{aligned}$$

Hence, optimists think a boom lasts on average 10 years whereas pessimists think it lasts for only three years (and vice versa for the recession).

C. Appendix: Omitted Derivations for the Welfare Analysis

This appendix presents the omitted derivations and proofs for the welfare analysis of the dynamic model that we present in Section 6. Section C.1 establishes the properties of the equilibrium value functions that are used in the main text. Section C.2 describes the details of the equilibrium with macroprudential policy, presents the analyses omitted from the main text (e.g., macroprudential policy in the high-risk-premium state), and presents omitted proofs.

C.1. Value functions in equilibrium

We first derive the HJB equation that describes the normalized value function in equilibrium and derive Eqs. (46). We then derive the differential equations in α-domain that characterize the value function and its components, and derive Eq. (49). We then prove Lemmas 1 and 2 that are used in the analysis.

Characterizing the normalized value function in equilibrium. Consider the recursive version of the portfolio problem in ($B.2$). Recall that the value function has the functional form in Eq. (45). Our goal is to characterize the value function per unit of capital, $v^i_{t,s}$ (corresponding to $a^i_{t,s} = Q_{t,s}$). To facilitate the analysis, we define,

$$\xi^i_{t,s} = v^i_{t,s} - \frac{\log Q_{t,s}}{\rho}. \tag{C.1}$$

Note that $\xi^i_{t,s}$ is the value function per unit wealth (corresponding to $a^i_{t,s} = 1$), and that the value function also satisfies $V^i_{t,s}\left(a^i_{t,s}\right) = \frac{\log\left(a^i_{t,s}\right)}{\rho} + \xi^i_{t,s}$. We first characterize $\xi^i_{t,s}$. We then combine this with Eq. ($C.1$) to characterize our main object of interest, $v^i_{t,s}$.

Consider the HJB equation ($B.2$). We substitute the optimal consumption rule from Eq. (22), the contingent allocation rule from Eq. (24), and $a^i_{t,s} = 1$ (to characterize the value per unit wealth) to obtain,

$$\begin{aligned}\rho\xi^i_{t,s} = \log\rho + \frac{1}{\rho}\left(r^f_{t,s} + \omega^{m,i}_{t,s}\left(r^m_{t,s} - r^f_{t,s}\right) - \frac{1}{2}\left(\omega^{m,i}_{t,s}\right)^2\sigma^2_s - \rho - \omega^{s',i}_{t,s}\right) \\ + \frac{\partial \xi^i_{t,s}}{\partial t} + \lambda^i_s\left(\frac{1}{\rho}\log\left(\frac{\lambda^i_s}{p^{s'}_{t,s}}\right) + \xi^i_{t,s'} - \xi^i_{t,s}\right).\end{aligned} \tag{C.2}$$

As we describe in Section 5, the market clearing conditions imply the optimal investment in the market portfolio and contingent securities satisfies, $\omega^m = 1$ and $\tilde{\omega}^{s',i}_{t,s} = \lambda^i_s - \overline{\lambda}_{t,s}$, and the price of the contingent security is given by, $p^{s'}_{t,s} = \overline{\lambda}_{t,s}\frac{1/Q_{t,s'}}{1/Q_{t,s}}$. Here, $\overline{\lambda}_{t,s}$ denotes the weighted average belief defined in (38). Using these conditions, the HJB equation becomes,

$$\begin{aligned}\rho\xi^i_{t,s} = \log\rho + \frac{1}{\rho}\begin{pmatrix} r^m_{t,s} - \rho - \frac{1}{2}\sigma^2_s \\ -\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^i_s}{\overline{\lambda}_{t,s}}\right)\end{pmatrix} \\ + \frac{\partial \xi^i_{t,s}}{\partial t} + \lambda^i_s\left(\frac{1}{\rho}\log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi^i_{t,s'} - \xi^i_{t,s}\right).\end{aligned} \tag{C.3}$$

After substituting the return to the market portfolio from (28), the HJB equation can be further simplified as,

$$\rho\xi_{t,s}^{i}=\begin{array}{c}\log\rho+\frac{1}{\rho}\left(\begin{array}{c}g+\mu_{t,s}^{Q}-\frac{1}{2}\sigma_{s}^{2}\\-\left(\lambda_{s}^{i}-\overline{\lambda}_{t,s}\right)+\lambda_{s}^{i}\log\left(\frac{\lambda_{s}^{i}}{\overline{\lambda}_{t,s}}\right)\end{array}\right)\\+\frac{\partial\xi_{t,s}^{i}}{\partial t}+\lambda_{s}^{i}\left(\frac{1}{\rho}\log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right)+\xi_{t,s'}^{i}-\xi_{t,s}^{i}\right)\end{array}.$$

Here, the term inside the summation on the second line, $-\left(\lambda_{s}^{i}-\overline{\lambda}_{t,s}\right)+\lambda_{s}^{i}\log\left(\frac{\lambda_{s}^{i}}{\overline{\lambda}_{t,s}}\right)$, is zero when there are no disagreements, and it is strictly positive when there are disagreements. This illustrates that speculation increases the expected value for optimists as well as pessimists.

We finally substitute $v_{t,s}^{i}=\xi_{t,s}^{i}+\frac{\log Q_{t,s}}{\rho}$ (cf. (C.1)) into the HJB equation to obtain the differential equation,

$$\rho v_{t,s}^{i}=\begin{array}{c}\log\rho+\log\left(Q_{t,s}\right)+\frac{1}{\rho}\left(\begin{array}{c}g-\frac{1}{2}\sigma_{s}^{2}\\-\left(\lambda_{s}^{i}-\overline{\lambda}_{t,s}\right)+\lambda_{s}^{i}\log\left(\frac{\lambda_{s}^{i}}{\overline{\lambda}_{t,s}}\right)\end{array}\right)\\+\frac{\partial v_{t,s}^{i}}{\partial t}+\lambda_{s}^{i}\left(v_{t,s'}-v_{t,s}\right)\end{array}.$$

Here, we have canceled terms by using the observation that $\frac{\partial\xi_{t,s}^{i}}{\partial t}=\frac{\partial v_{t,s}^{i}}{\partial t}-\frac{1}{\rho}\frac{\partial\log Q_{t,s}}{\partial t}=\frac{\partial v_{t,s}^{i}}{\partial t}-\frac{1}{\rho}\mu_{t,s}^{Q}$. We have thus obtained Eq. (46) in the main text.

Differential equations for the value functions in α-domain. The value function and its components, $\left\{v_{t,s}^{i},v_{t,s}^{i,*},w_{t,s}\right\}_{s,i}$, can be written as functions of optimists' wealth share, $\left\{v_{s}^{i}\left(\alpha\right),v_{s}^{i,*}\left(\alpha\right),w_{s}\left(\alpha\right)\right\}_{s,i}$, that solve appropriate ordinary differential equations. We next represent the value functions as solutions to the differential equations in α-domain. Recall that the price level in each state can be written as a function of optimists' wealth shares, $q_{t,s}=q_{s}\left(\alpha\right)$ (where we also have, $q_{1}\left(\alpha\right)=q^{*}$). Plugging in these price functions, and using the dynamics of $\alpha_{t,s}$ from Eq. (41), the HJB equation (46) can be written as,

$$\rho v_{s}^{i}\left(\alpha\right)=\begin{array}{c}\log\rho+q_{s}\left(\alpha\right)+\frac{1}{\rho}\left(\begin{array}{c}g-\frac{1}{2}\sigma_{s}^{2}\\-\left(\lambda_{s}^{i}-\overline{\lambda}_{s}\left(\alpha\right)\right)+\lambda_{s}^{i}\log\left(\frac{\lambda_{s}^{i}}{\overline{\lambda}_{s}\left(\alpha\right)}\right)\end{array}\right)\\-\frac{\partial v_{s}^{i}}{\partial\alpha}\left(\lambda_{s}^{o}-\lambda_{s}^{p}\right)\alpha\left(1-\alpha\right)+\lambda_{s}^{i}\left(v_{s'}^{i}\left(\alpha\frac{\lambda_{s}^{o}}{\overline{\lambda}_{s}\left(\alpha\right)}\right)-v_{s}^{i}\left(\alpha\right)\right)\end{array}.$$

For each $i\in\{o,p\}$, the value functions, $\left(v_{s}^{i}\left(\alpha\right)\right)_{s\in\{1,2\}}$, are found by solving this system of ODEs. For $i=o$, the boundary conditions are that the values, $\left\{v_{s}^{o}\left(1\right)\right\}_{s}$, are the same as the values in the common belief benchmark characterized in Section 4 when all investors have the optimistic beliefs. For $i=p$, the boundary conditions are that the values, $\left\{v_{s}^{p}\left(0\right)\right\}_{s}$, are the same as the values in the common belief benchmark when all investors have the pessimistic beliefs.

Likewise, the first-best value functions, $\left(v_{s}^{i,*}\left(\alpha\right)\right)_{s\in\{1,2\}}$, are found by solving the analogous system after replacing $q_{s}\left(\alpha\right)$ with q^{*} (and changing the boundary conditions appropriately). Finally, substituting the price functions into Eq. (48), the gap-value functions, $\left(w_{s}^{i}\left(\alpha\right)\right)_{s,i}$, are found by solving the system in (49).

For the proofs in this section (as well as in some subsequent sections), we find it useful to work with the transformed state variable,

$$b_{t,s}\equiv\log\left(\frac{\alpha_{t,s}}{1-\alpha_{t,s}}\right),\text{ which implies }\alpha_{t,s}=\frac{1}{1+\exp\left(-b_{t,s}\right)}.\tag{C.4}$$

The variable, $b_{t,s}$, varies between $(-\infty,\infty)$ and provides a different measure of optimism, which we refer to as "bullishness." Note that there is a one-to-one relation between optimists' wealth share, $\alpha_{t,s}\in(0,1)$, and the bullishness, $b_{t,s} \in \mathbb{R} = (-\infty, +\infty)$. Optimists' wealth dynamics in (41) become particularly simple when expressed in terms of bullishness,

$$\begin{cases} \dot{b}_{t,s} = -\left(\lambda_s^o - \lambda_s^p\right), & \text{if there is no state change,} \\ b_{t,s'} = b_{t,s} + \log \lambda_s^o - \log \lambda_s^p, & \text{if there is a state change.} \end{cases} \tag{C.5}$$

With a slight abuse of notation, we also let $q_2(b), w_s^i(b)$, and so on, denote the equilibrium functions in terms of bullishness. Note also that, since $\frac{db}{d\alpha} = \frac{1}{\alpha(1-\alpha)}$, we have the identities,

$$\frac{\partial q_2(b)}{\partial b} = \alpha(1-\alpha)\frac{\partial q_2(\alpha)}{\partial b} \text{ and } \frac{\partial w_s^i(b)}{\partial b} = \alpha(1-\alpha)\frac{\partial w_s^i(\alpha)}{\partial \alpha}. \tag{C.6}$$

Using this observation, the differential equation for the price function, Eq. (43), can be written in terms of bullishness as,

$$\frac{\partial q_2(b)}{\partial b}\left(\lambda_2^o - \lambda_2^p\right) = \rho + g + \overline{\lambda}_2(\alpha)\left(1 - \frac{Q_2}{Q^*}\right) - \sigma_2^2. \tag{C.7}$$

Likewise, the differential equation for the gap value function, Eq. (49), can be written in terms of bullishness as,

$$\rho w_s^i(b) = q_s(b) - q^* - \left(\lambda_s^o - \lambda_s^p\right)\frac{\partial w_s^i(b)}{\partial b} + \lambda_s^i\left(w_{s'}^i(b') - w_s^i(b)\right). \tag{C.8}$$

Proof of Lemma 1. To show that the gap value function is increasing, consider its representation in terms of bullishness, $w_s^i(b)$ [cf. ($C.4$)], which solves the system in ($C.8$). We will first describe this function as a fixed point of a contraction mapping. We will then use this contraction mapping to establish the properties of the function.

Recall that, in the time domain, the gap value function solves the HJB equation (48). Integrating this equation forward, we obtain,

$$w_s^i(b_{0,s}) = \int_0^\infty e^{-\left(\rho+\lambda_s^i\right)t}\left(q_s(b_{t,s}) - q^* + \lambda_s^i w_{s'}^i(b_{t,s'})\right)dt, \tag{C.9}$$

for each $s \in \{1,2\}$ and $b_{0,s} \in \mathbb{R}$. Here, $b_{t,s}$ denotes bullishness conditional on there not being a transition before time t, whereas $b_{t,s'}$ denotes the bullishness if there is a transition at time t. Solving Eq. ($C.5$) (given beliefs, λ^i) we further obtain,

$$\begin{aligned} b_{t,s} &= b_{0,s} - t\left(\lambda_s^o - \lambda_s^p\right), \\ b_{t,s'} &= b_{0,s} - t\left(\lambda_s^o - \lambda_s^p\right) + \log \lambda_s^o - \log \lambda_s^p. \end{aligned} \tag{C.10}$$

Hence, Eq. ($C.9$) describes the value function as a solution to an integral equation given the closed form solution for bullishness in ($C.10$).

Implicitly differentiating the integral equation ($C.9$) with respect to $b_{0,s}$, and using Eq. ($C.10$), we also obtain,

$$\frac{\partial w_s^i(b_{0,s})}{\partial b} = \int_0^\infty e^{-\left(\rho+\lambda_s^i\right)t}\left(\frac{\partial q_s(b_{t,s})}{\partial b} + \lambda_s^i\frac{\partial w_{s'}^i(b_{t,s'})}{\partial b}\right)dt. \tag{C.11}$$

We next let $B\left(\mathbb{R}^2\right)$ denote the set of bounded value functions over $\mathbb{R}^2$. Given some continuation value function, $\left(\frac{\partial \tilde{w}_s^i(b)}{\partial b}\right)_s \in B\left(\mathbb{R}^2\right)$, we define the function, $\left(T\frac{\partial \tilde{w}_s^i(b)}{\partial b}\right)_s \in B\left(\mathbb{R}^2\right)$, so that

$$T\frac{\partial \tilde{w}_s^i\left(b_{0,s}\right)}{\partial b} = \int_0^\infty e^{-\left(\rho+\lambda_s^i\right)t}\left(\frac{\partial q_s\left(b_{t,s}\right)}{\partial b} + \lambda_s^i \frac{\partial \tilde{w}_{s'}^i\left(b_{t,s'}\right)}{\partial b}\right)dt,$$

for each s and $b_{0,s} \in \mathbb{R}$. Note also that the resulting value functions are bounded since the derivative of the price functions, $\left(\frac{\partial q_s(b_{t,s})}{\partial b}\right)_s$, are bounded (see Eq. ($C.7$)). Thus, Eq. ($C.11$) describes the derivative functions, $\left(\frac{\partial w_s^i(b_{0,s})}{\partial b}\right)_s$, as a fixed point of a corresponding operator T over bounded functions. It can be checked that this operator is a contraction mapping with respect to the sup norm. Thus, it has a unique fixed point that corresponds to the derivative functions. Moreover, since $\frac{\partial q_s(b_{t,s})}{\partial b} > 0$ for each b, and $\lambda_s^i > 0$ for each s, it can further be seen that the fixed point satisfies, $\frac{\partial w_s^i(b_{0,s})}{\partial b} > 0$ for each b and $s \in \{1, 2\}$. Using Eq. ($C.6$), we also obtain $\frac{\partial w_s^i(\alpha)}{\partial \alpha} > 0$ for each $\alpha \in (0, 1)$ and $s \in \{1, 2\}$, completing the proof. □

Proof of Lemma 2. Consider the analysis in Lemma 1 for the special case, $\lambda_1^o = \lambda_1^p$. Applying Eq. ($C.11$) for $s = 1$, we obtain [since $q_1\left(b_{t,s}\right) = q^*$ is constant],

$$\frac{\partial w_1^i\left(b_{0,1}\right)}{\partial b} = \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_1^i \frac{\partial w_2^i\left(b_{t,2}\right)}{\partial b}dt.$$

Note also that $\lambda_1^o = \lambda_1^p$ and Eq. ($C.10$) imply $b_{t,2} = b_{0,1}$ (since there is no speculation). Substituting this into the displayed equation, we obtain $\frac{\partial w_1^i(b_{0,1})}{\partial b} = \frac{\lambda_1^i}{\rho+\lambda_1^i}\frac{\partial w_2^i(b_{0,1})}{\partial b} < \frac{\partial w_2^i(b_{0,1})}{\partial b}$. Combining this with Eq. ($C.6$) completes the proof. □

C.2. Equilibrium with macroprudential policy

Recall that macroprudential policy induces optimists to choose allocations as if they have more pessimistic beliefs, $\lambda^{o,pl} \equiv \left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, that satisfy, $\lambda_1^{o,pl} \geq \lambda_1^o$ and $\lambda_2^{o,pl} \leq \lambda_2^o$. We next show that this allocation can be implemented with portfolio restrictions on optimists. We then show that the planner's Pareto problem reduces to solving problem (51) in the main text. We also derive the equilibrium value functions that result from macroprudential policy. We then analyze macroprudential policy in the recession state, which complements the analysis in the main text (that focuses on the boom state), and present Proposition 4. Finally, we present the proofs of Propositions 3 and 4.

Implementing the policy with risk limits. Consider the equilibrium that would obtain if optimists had the planner-induced beliefs, $\lambda_s^{o,pl}$. Using our analysis in Section 5, optimists' equilibrium portfolios are given by,

$$\omega_{t,s}^{m,o,pl} = 1 \text{ and } \omega_{t,s}^{s',o,pl} = \lambda_s^{o,pl} - \overline{\lambda}_{t,s}^{pl} \text{ for each } t, s. \tag{C.12}$$

We first show that the planner can implement the policy by requiring optimists to hold exactly these portfolio weights. We will then relax these portfolio constraints into inequality restrictions (see Eq. ($C.14$)).

Formally, an optimist solves the HJB problem ($B.2$) with the additional constraint ($C.12$). In view of log utility, we conjecture that the value function has the same functional form (45) with potentially different normalized values, $\xi_{t,s}^o, v_{t,s}^o$, that reflect the constraints. Using this functional form, the optimality condition for consumption remains unchanged, $c_{t,s} = \rho a_{t,s}^o$ [cf. Eq. (22)]. Plugging this equation and the portfolio holdings in ($C.12$) into the objective function in ($B.2$) verifies that the value function has the conjectured functional form. For later reference, we also obtain that the optimists' unit-wealth value function satisfies [cf. Eq. (C.1)],

$$
\begin{aligned}
\xi^o_{t,s} &= \log\rho + \frac{1}{\rho}\left(r^f_{t,s} + \omega^{m,o,pl}_{t,s}\left(r^m_{t,s} - r^f_{t,s}\right) - \rho - \omega^{s',o,pl}_{t,s}\right) \\
&\quad -\frac{1}{2\rho}\left(\omega^{m,o,pl}_{t,s}\sigma_s\right)^2 + \frac{\partial \xi^o_{t,s}}{\partial t} + \lambda^o_s\left(\frac{1}{\rho}\log\left(\frac{a^o_{t,s'}}{a^o_{t,s}}\right) + \xi^o_{t,s'} - \xi^o_{t,s}\right).
\end{aligned}
\tag{C.13}
$$

Here, $\frac{a^o_{t,s'}}{a^o_{t,s}} = 1 + \omega^{m,o,pl}_{t,s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}} + \frac{\omega^{s',o,pl}_{t,s}}{p^{s'}_{t,s}}$ in view of the budget constraints (B.1). Hence, the value function has a similar characterization as before [cf. Eq. (C.2)] with the difference that optimists' portfolio holdings reflect the portfolio constraints.

Since pessimists are unconstrained, their optimality conditions are unchanged. It follows that the equilibrium takes the form in Section 5 with the difference that investors' beliefs are replaced by their as-if beliefs, $\lambda^{i,pl}_s$. This verifies that the planner can implement the policy using the portfolio restrictions in (C.12). We next show that these restrictions can be relaxed to the following inequality constraints,

$$
\begin{aligned}
\omega^{m,o,pl}_{t,s} &\leq 1 \text{ for each } s, \\
\omega^{2,o,pl}_{t,1} &\geq \underline{\omega}^{2,o}_{t,1} \equiv \lambda^{o,pl}_1 - \overline{\lambda}^{pl}_{t,1} \text{ and } \omega^{1,o,pl}_{t,2} \leq \overline{\omega}^{1,o}_{t,2} \equiv \lambda^{o,pl}_2 - \overline{\lambda}^{pl}_{t,2}.
\end{aligned}
\tag{C.14}
$$

In particular, we will establish that all inequality constraints bind, which implies that optimists optimally choose the portfolio weights in Eq. (C.12). Thus, our earlier analysis continues to apply when optimists are subject to the more relaxed restrictions in (C.14).

The result follows from the assumption that the planner-induced beliefs are more pessimistic than optimists' actual beliefs, $\lambda^{o,pl}_1 \geq \lambda^o_1$ and $\lambda^{o,pl}_2 \leq \lambda^o_2$. To see this formally, note that the optimality condition for the market portfolio is given by the following generalization of Eq. (23),

$$
\omega^{m,o,pl}_{t,s}\sigma_s \leq \frac{1}{\sigma_s}\left(r^m_{t,s} - r^f_{t,s} + \lambda^o_s\frac{a^o_{t,s}}{a^o_{t,s'}}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right) \text{ and } \omega^{m,o,pl}_{t,s} \leq 1,
\tag{C.15}
$$

with complementary slackness. Note also that,

$$
\lambda^o_s\frac{a^o_{t,s}}{a^o_{t,s'}}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}} = \lambda^o_s\frac{\overline{\lambda}^{pl}_{t,s}}{\lambda^{o,pl}_s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}} \geq \overline{\lambda}^{pl}_{t,s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}} \text{ for each } s.
$$

Here, the equality follows because Eq. (B.12) in Appendix B.3 applies with as-if beliefs. The inequality follows by considering separately the two cases, $s \in \{1,2\}$. For $s = 2$, the inequality holds since $Q_{t,s'} - Q_{t,s} > 0$ and the beliefs satisfy, $\lambda^o_s \geq \lambda^{o,pl}_s$. For $s = 1$, the inequality holds since $Q_{t,s'} - Q_{t,s} < 0$ and the beliefs satisfy, $\lambda^{o,pl}_s \geq \lambda^o_s$. Note also that in equilibrium the return to the market portfolio satisfies Eq. (B.9), which we replicate here,

$$
\sigma_s = \frac{1}{\sigma_s}\left(r^m_{t,s} - r^f_{t,s} + \overline{\lambda}^{pl}_{t,s}\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right)\right).
$$

Combining these expressions implies, $\sigma_s \leq \frac{1}{\sigma_s}\left(r^m_{t,s} - r^f_{t,s} + \lambda^o_s\frac{a^o_{t,s}}{a^o_{t,s'}}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right)$, which in turn implies the optimality condition (C.15) is satisfied with $\omega^{m,o,pl}_{t,s} = 1$. A similar analysis shows that optimists also choose the corner allocations in contingent securities, $\omega^{2,o,pl}_{t,1} = \underline{\omega}^{2,o}_{t,1}$ and $\omega^{1,o,pl}_{t,2} = \overline{\omega}^{1,o}_{t,2}$, verifying that the portfolio constraints (C.12) can be relaxed to the inequality constraints in (C.14).

Simplifying the planner's problem. Recall that, to trace the Pareto frontier, we allow the planner to do a one-time wealth transfer among the investors at time 0. Let $V_{t,s}^{i}\left(a_{t,s}^{i}|\left\{\lambda_{t}^{o,pl}\right\}\right)$ denote type i investors' expected value in equilibrium when she starts with wealth $a_{t,s}^{i}$ and the planner commits to implement the policy, $\left\{\lambda_{t}^{o,pl}\right\}$. Then, the planner's Pareto problem can be written as,

$$\max_{\tilde{\lambda}^{o,pl},\tilde{\alpha}_{0,s}} \gamma^{o} V_{0,s}^{o}\left(\tilde{\alpha}_{0,s} Q_{0,s} k_{0,s} | \tilde{\lambda}^{o,pl}\right)+\gamma^{p} V_{0,s}^{p}\left(\left(1-\tilde{\alpha}_{0,s}\right) Q_{0,s} k_{0,s} | \tilde{\lambda}^{o,pl}\right). \tag{C.16}$$

Here, $\gamma^{o},\gamma^{p} \geq 0$ (with at least one strict inequality) denote the Pareto weights, and $Q_{0,s}$ denotes the endogenous equilibrium price that obtains under the planner's policy.

Next recall that the investors' value function with macroprudential policy has the same functional form in (45) (with potentially different $\xi_{t,s}^{o}, v_{t,s}^{o}$ for optimists that reflect the constraints). After substituting $a_{t,s}^{i}=\alpha_{t,s}^{i} k_{t,s} Q_{t,s}$, the functional form implies,

$$V_{t,s}^{i}=v_{t,s}^{i}+\frac{\log\left(\alpha_{t,s}^{i}\right)+\log\left(k_{t,s}\right)}{\rho}.$$

Using this expression, the planner's problem ($C.16$) can be rewritten as,

$$\max_{\tilde{\lambda}^{o,pl},\tilde{\alpha}_{0,s}}\left(\gamma^{o} v_{0,s}^{o}+\gamma^{p} v_{0,s}^{p}\right)+\frac{\gamma^{o} \log\left(\tilde{\alpha}_{0,s}^{o}\right)+\gamma^{p} \log\left(1-\tilde{\alpha}_{0,s}^{o}\right)}{\rho}+\frac{\left(\gamma^{o}+\gamma^{p}\right) \log\left(k_{0,s}\right)}{\rho}.$$

Here, the last term (that features capital) is a constant that doesn't affect optimization. The second term links the planner's choice of wealth redistribution, $\alpha_{0,s}^{o},\alpha_{0,s}^{p}$, to her Pareto weights, γ^{o},γ^{p}. Specifically, the first order condition with respect to optimists' wealth share implies $\frac{\gamma^{o}}{\gamma^{p}}=\frac{\alpha_{0,s}}{1-\alpha_{0,s}}$. Thus, the planner effectively maximizes the first term after substituting γ^{o} and γ^{p} respectively with the optimal choice of $\alpha_{0,s}$ and $1-\alpha_{0,s}$. This leads to the simplified problem (51) in the main text.

Characterizing the value functions with macroprudential policy. We first show that the normalized value functions, $v_{t,s}^{i}$, are characterized as the solution to the following differential equation system,

$$\rho v_{t,s}^{i}-\frac{\partial v_{t,s}^{i}}{\partial t}=\log \rho+q_{t,s}+\frac{1}{\rho}\left(\begin{array}{c} g-\frac{1}{2}\sigma_{s}^{2} \\ -\left(\lambda_{s}^{i,pl}-\overline{\lambda}_{t,s}^{pl}\right)+\lambda_{s}^{i}\log\left(\frac{\lambda_{s}^{i,pl}}{\overline{\lambda}_{t,s}^{pl}}\right)\end{array}\right)+\lambda_{s}^{i}\left(v_{t,s'}^{i}-v_{t,s}^{i}\right). \tag{C.17}$$

This is a generalization of Eq. (46) in which investors' positions are calculated according to their as-if beliefs, $\lambda_{s}^{i,pl}$, but the transition probabilities are calculated according to their actual beliefs, λ_{s}^{i}.

First consider the pessimists. Since they are unconstrained, their value function is characterized by solving the earlier equation system ($C.13$). In this case, equation ($C.17$) also holds since it is the same as the earlier equation.

Next consider the optimists. In this case, the analysis in Appendix B.3 applies with as-if beliefs. In particular, we have [cf. Eqs. ($B.12$) and ($B.13$)],

$$\frac{a_{t,s'}^{o}}{a_{t,s}^{o}}=\frac{\lambda_{s}^{o,pl}}{\overline{\lambda}_{t,s}^{pl}}\frac{Q_{t,s'}}{Q_{t,s}}.$$

Plugging this expression as well as Eq. ($C.12$) into Eq. ($C.13$), optimists' unit-wealth value function satisfies,

$$\begin{aligned} \xi^o_{t,s} &= \log\rho + \frac{1}{\rho}\left(\begin{array}{c} r^m_{t,s} - \rho - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda_s^{o,pl} - \overline{\lambda}^{pl}_{t,s}\right) + \lambda^o_s \log\left(\frac{\lambda^{o,pl}_{t,s}}{\overline{\lambda}^{pl}_{t,s}}\right) \end{array}\right) \\ &\quad + \frac{\partial \xi^o_{t,s}}{\partial t} + \lambda^o_s\left(\frac{1}{\rho}\log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi^o_{t,s'} - \xi^o_{t,s}\right), \end{aligned}$$

This is the same as Eq. ($C.13$) with the difference that the as-if beliefs, $\lambda_s^{o,pl}$, are used to calculate their positions on (and the payoffs from) the contingent securities, whereas the actual beliefs, λ^o_s, are used to calculate the transition probabilities. Using the same steps after Eq. ($C.13$), we also obtain ($C.17$) with $i = o$.

We next characterize the first-best and the gap value functions, $v^{i,*}_{t,s}$ and $w^i_{t,s}$, that we use in the main text. By definition, the first-best value function solves the same differential equation ($C.17$) after substituting $q_{t,s} = q^*$. It follows that the gap value function $w^i_{t,s} = v^i_{t,s} - v^{i,*}_{t,s}$, solves,

$$\rho w^i_{t,s} - \frac{\partial w^i_{t,s}}{\partial t} = q_{t,s} - q^* + \lambda^i_s\left(w^i_{t,s'} - w^i_{t,s}\right),$$

which is the same as the differential equation (48) without macroprudential policy. The latter affects the path of prices, $q_{t,s}$, but it does not affect how these prices translate into gap values.

Note also that, as before, the value functions can be written as functions of optimists' wealth share, $\left\{v^i_s(\alpha), v^{i,*}_s(\alpha), w_s(\alpha)\right\}_{s,i}$. For completeness, we also characterize the differential equations that these functions satisfy in equilibrium with macroprudential policy. Combining Eq. ($C.17$) with the dynamics of optimists' wealth share conditional on no transition, $\dot{\alpha}_{t,s} = -\left(\lambda^{o,pl}_s - \lambda^p_s\right)\alpha_{t,s}(1-\alpha_{t,s})$, the value functions, $\left(v^i_s(\alpha)\right)_{s,i}$, are found by solving,

$$\rho v^i_s(\alpha) = \left[\begin{array}{c} \log\rho + q^{pl}_s(\alpha) + \frac{1}{\rho}\left(\begin{array}{c} g - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda^{i,pl}_s - \overline{\lambda}^{pl}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^{i,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right)\end{array}\right) \\ -\frac{\partial v^i_s}{\partial \alpha}\left(\lambda^{o,pl}_s - \lambda^p_s\right)\alpha(1-\alpha) + \lambda^i_s\left(v^i_{s'}\left(\alpha\frac{\lambda^{o,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right) - v^i_s(\alpha)\right)\end{array}\right],$$

with appropriate boundary conditions. As in the main text, we denote the price functions with $q^{pl}_s(\alpha)$ to emphasize that they are determined by as-if beliefs. Likewise, the first-best value functions, $\left(v^{i,*}_s(\alpha)\right)_{s\in\{1,2\}}$, are found by solving the analogous system after replacing $q_s(\alpha)$ with q^*. Finally, combining Eq. (48) with the dynamics of optimists' wealth share, the gap-value functions, $\left(w^i_s(\alpha)\right)_{s,i}$, are found by solving Eq. (50) in the main text

Macroprudential policy in the recession state. The analysis in the main text concerns macroprudential policy in the boom state and maintains the assumption that $\lambda^{o,pl}_2 = \lambda^o_2$. We next consider the polar opposite case in which the economy is currently in the recession state $s = 2$, and the planner can apply macroprudential policy in this state, $\lambda^{o,pl}_2 \leq \lambda^o_2$ (she can induce optimists to act as if the recovery is less likely), but not in the other state, $\lambda^{o,pl}_1 = \lambda^o_1$. We obtain a sharp result for the special case in which optimists' wealth share is sufficiently large.

Proposition 4. *Consider the model with two belief types. Consider the macroprudential policy in the recession state, $\lambda_2^{o,pl} \leq \lambda_2^o$ (and suppose $\lambda_1^{o,pl} = \lambda_1^o$). There exists a threshold, $\overline{\alpha} < 1$, such that if $\alpha \in (\overline{\alpha}, 1]$, then the policy reduces the gap value according to each belief, that is,*

$$\left. \frac{\partial w_2^i(\alpha)}{\partial \left(-\lambda_2^{o,pl}\right)} \right|_{\lambda_2^{o,pl} = \lambda_2^o} < 0 \textit{ for each } i \in \{o, p\}.$$

Thus, for $\alpha \in (\overline{\alpha}, 1]$, the policy also reduces the planner's value, $\left. \frac{\partial v_s^{pl}(\alpha)}{\partial(-\lambda_2^{o,pl})} \right|_{\lambda_2^o} = \left. \frac{\partial w_s^{pl}(\alpha)}{\partial(-\lambda_2^{o,pl})} \right|_{\lambda_2^o} < 0$.

Thus, in contrast to Proposition 3, macroprudential policy in the recession state can actually reduce the gap value (and therefore also the social welfare). The intuition can be understood by considering two counteracting forces. First, as before, macroprudential policy in the recession state is potentially valuable by reallocating optimists' wealth from the boom state $s = 1$ to the recession state $s = 2$. Intuitively, optimists purchase too many call options that pay if there is a transition to the boom state but that impoverish them in case the recession persists. They do not internalize that, if they keep their wealth, they will improve asset prices if the recession lasts longer.

However, there is a second force that does not have a counterpart in the boom state: Macroprudential policy in the recession state also affects the current asset price level, with potential implications for gap value. It can be seen that making optimists less optimistic in the recession state shifts the price function downward, $\frac{\partial q_2^{pl}(\alpha)}{\partial(-\lambda_2^{o,pl})} < 0$ (as in the common-belief benchmark we analyzed in Section 4). Hence, the price impact of macroprudential policy reduces the gap value. Moreover, as optimists dominate the economy, $\alpha \to 1$, the price impact of the policy is still first order, whereas the beneficial effect from reshuffling optimists' wealth is second order. Thus, when optimists' wealth share is sufficiently large, the net effect of macroprudential policy on the gap value is negative.

This analysis also suggests that, even when the policy in the recession state exerts a net positive effect, it would typically increase the gap value by a smaller amount than a comparable policy in the boom state. Figure 9 confirms this intuition. The left panel plots the change in the planner's gap value function in the boom state resulting from a small macroprudential policy change. Note that the policy slightly reduces the planner's first-best value function but increases the gap value function. The right panel illustrates the effect of the macroprudential policy in the recession state that would generate a similar distortion in the first-best equilibrium as the policy in the boom state.[30] Note that a small macroprudential policy in the recession state has a smaller positive impact on the gap value when optimists' wealth share is small, and it has a negative impact when optimists' wealth share is sufficiently large, illustrating Proposition 4.

Proof of Proposition 3. We will prove the stronger result that

$$\left. \frac{\partial w_s^i(\alpha)}{\partial \lambda_1^{o,pl}} \right|_{\lambda^{o,pl} = \lambda^o} > 0 \text{ for each } i, s \text{ and } \alpha \in (0, 1). \tag{C.18}$$

That is, a marginal amount of macroprudential policy in the low-risk-premium state increases the gap value according to each investor (and in either state). Combining this with the definition of the planner's gap value function in (52) implies $\left. \frac{\partial w_s^{pl}(\alpha)}{\partial \lambda_1^{o,pl}} \right|_{\lambda^{o,pl} = \lambda^o} > 0$. Combining this with $\left. \frac{\partial v_{0,s}^{pl,*}}{\partial \lambda^{o,pl}} \right|_{\lambda^{o,pl} = \lambda^o} = 0$ (which follows

[30] Specifically, we calibrate the policy-induced belief change in the recession state so that the maximum decline in the planner's first-best value function is the same in both cases plotted in Figure 9, $\max_\alpha \left| \Delta v_2^{pl,*}(\alpha) \right| = \max_\alpha \left| \Delta v_1^{pl,*}(\alpha) \right|$.

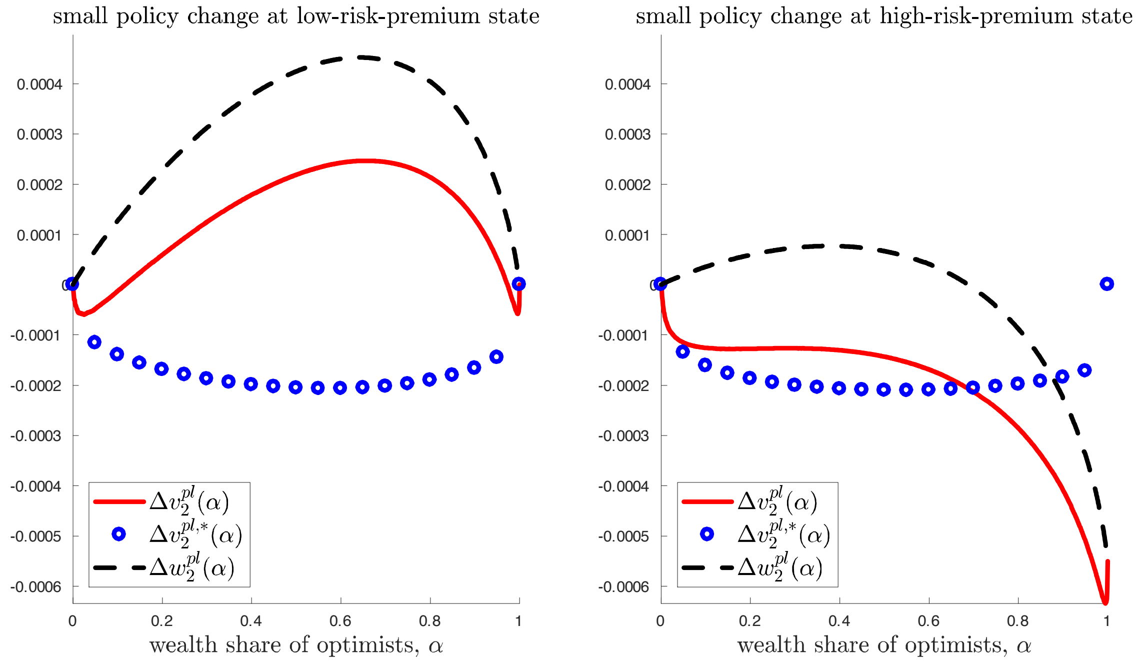

Figure 9: The left (resp. the right) panel illustrates the effect of a small change in macroprudential policy in the boom (resp. the recession) state.

from the First Welfare Theorem) and $v_{0,s}^{pl} = v_{0,s}^{pl,*} + w_{0,s}^{pl}$ implies $\left.\frac{\partial w_s^{pl}(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^o} = \left.\frac{\partial v_s^{pl}(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^o}$ for each s and $\alpha \in (0,1)$. Applying this result for state $s = 1$ proves the proposition.

It remains to prove the claim in ($C.18$). To this end, fix a belief type i and consider the representation of the gap value function in terms of bullishness, $w_s^i(b)$ [cf. ($C.4$)]. Following similar steps as in Lemma 1, we describe this as solution to the integral function,

$$w_s^i(b_{0,s}) = \int_0^\infty e^{-\left(\rho+\lambda_s^i\right)t}\left(q_s^{pl}(b_{t,s}) - q^* + \lambda_s^i w_{s'}^i(b_{t,s'})\right)dt, \tag{C.19}$$

for each $s \in \{1,2\}$ and $b_{0,s} \in \mathbb{R}$, where the bullishness has the closed form solution,

$$\begin{aligned} b_{t,s} &= b_{0,s} - t\left(\lambda_s^{o,pl} - \lambda_s^p\right), \\ b_{t,s'} &= b_{0,s} - t\left(\lambda_s^{o,pl} - \lambda_s^p\right) + \log\lambda_s^{o,pl} - \log\lambda_s^p. \end{aligned} \tag{C.20}$$

The main difference from the analysis in Lemma 1 is that the dynamics of bullishness is influenced by policy, as illustrated by the as-if beliefs in ($C.10$). In addition, we denote the price functions with $q_s^{pl}(b)$ to emphasize they are in principle determined by as-if beliefs.

Next note that in this case the price functions $q_s^{pl}(b)$ are actually not affected by the as-if belief, $\lambda_1^{o,pl}$. The price function in the low-risk-premium state is not affected because $q_1^{pl}(b) = q^*$ (because the beliefs continue to satisfy Assumption 3 for small changes). The price function in the high-risk-premium state is also not affected because $\lambda_1^{o,pl}$ does not enter the differential equation that characterizes $q_2^{pl}(b)$ [see. Eq. (43) or Eq. ($C.7$)].

Using this observation, we implicitly differentiate the integral equation ($C.19$) with respect to $\lambda_1^{o,pl}$, and use Eq. ($C.20$), to obtain,

$$\begin{aligned}\frac{\partial w_1^i\left(b_{0,1}\right)}{\partial\lambda_1^{o,pl}} &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_1^i\left(\frac{\partial w_2^i\left(b_{t,2}\right)}{\partial\lambda_1^{o,pl}}+\frac{\partial w_2^i\left(b_{t,2}\right)}{\partial b}\frac{db_{t,2}}{d\lambda_1^{o,pl}}\right)dt,\\ \frac{\partial w_2^i\left(b_{0,2}\right)}{\partial\lambda_1^{o,pl}} &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_2^i\frac{\partial w_1^i\left(b_{t,1}\right)}{\partial\lambda_1^{o,pl}}dt.\end{aligned}$$

Note also that, using Eq. ($C.20$) implies, $\frac{db_{t,2}}{d\lambda_1^{o,pl}} = -t+\frac{1}{\lambda_1^{o,pl}}$. Plugging this into the previous system, and evaluating the partial derivatives at $\lambda_1^{o,pl}=\lambda_1^o$, we obtain,

$$\begin{aligned}\frac{\partial w_1^i\left(b_{0,1}\right)}{\partial\lambda_1^{o,pl}} &= h\left(b_{0,1}\right)+\int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_1^i\frac{\partial w_2^i\left(b_{t,2}\right)}{\partial\lambda_1^{o,pl}}dt,\\ \frac{\partial w_2^i\left(b_{0,2}\right)}{\partial\lambda_1^{o,pl}} &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_2^i\frac{\partial w_1^i\left(b_{t,1}\right)}{\partial\lambda_1^{o,pl}}dt,\\ \text{where } h\left(b_{0,1}\right) &= \int_0^\infty e^{-\left(\rho+\lambda_1^i\right)t}\lambda_1^i\frac{\partial w_2^i\left(b_{t,2}\right)}{\partial b}\left(-t+\frac{1}{\lambda_1^o}\right)dt.\end{aligned} \tag{C.21}$$

Note that the function, $h(b)$, is bounded since the derivative function, $\frac{\partial w_2^i(b)}{\partial b}$, is bounded (see ($C.11$)). Hence, Eq. ($C.21$) describes the partial derivative functions, $\left(\frac{\partial w_s^i(b)}{\partial\lambda_1^{o,pl}}|_{\lambda_1^{o,pl}=\lambda_1^o}\right)_s$, as a fixed point of a corresponding operator T over bounded functions. Since $h(b)$ is bounded, it can be checked that the operator T is also a contraction mapping with respect to the sup norm. In particular, it has a fixed point, which corresponds to the partial derivative functions.

The analysis so far applies generally. We next consider the special case, $\lambda_1^o=\lambda_1^p$, and show that it implies the partial derivatives are strictly positive. In this case, $\lambda_1^i\equiv\lambda_1$ for each $i\in\{o,p\}$. In addition, Eq. ($C.10$) implies $b_{t,2}=b_{0,2}$. Using these observations, for each $b_{0,1}$, we have,

$$\begin{aligned}h\left(b_{0,1}\right) &= \frac{\partial w_2^i\left(b_{0,2}\right)}{\partial b}\int_0^\infty e^{-(\rho+\lambda_1)t}\lambda_1\left(-t+\frac{1}{\lambda_1}\right)dt\\ &= \frac{\partial w_2^i\left(b_{0,2}\right)}{\partial b}\left(-\frac{\lambda_1}{\rho+\lambda_1}\frac{1}{\rho+\lambda_1}+\frac{1}{\rho+\lambda_1}\right)>0.\end{aligned}$$

Here, the inequality follows since $\frac{\partial w_2^i(b_{0,2})}{\partial b}>0$ [cf. Lemma 1]. Since $h(b)>0$ for each b, and $\lambda_s^i>0$, it can further be seen that the fixed point that solves ($C.21$) satisfies $\frac{\partial w_s^i(b)}{\partial\lambda_s^{o,pl}}>0$ for each b and $s\in\{1,2\}$. Using Eq. ($C.6$), we also obtain $\frac{\partial w_s^i(\alpha)}{\partial\lambda_1^{o,pl}}>0$ for each $s\in\{1,2\}$ and $\alpha\in(0,1)$. Since the analysis applies for any fixed belief type i, this establishes the claim in ($C.18$) and completes the proof. $\square$

Proof of Proposition 4. A similar analysis as in the proof of Proposition 3 implies that the partial derivative function, $\frac{\partial w_s^i(b)}{\partial\left(-\lambda_2^{o,pl}\right)}$, is characterized as the fixed point of a contraction mapping over bounded functions (the analogue of Eq. ($C.21$) for state 2). In particular, the partial derivative exists and it is bounded. Moreover, since the corresponding contraction mapping takes continuous functions into continuous functions, the partial derivative is also continuous over $b\in\mathbb{R}$. Using Eq. ($C.6$), we further obtain that the partial derivative, $\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}$, is continuous over $\alpha\in(0,1)$.

Next note that $w_s^i(1)\equiv\lim_{\alpha\to1}w_s^i(\alpha)$ exists and is equal to the value function according to type i beliefs when all investors are optimistic. In particular, the asset prices are given by $q_1^{pl}=q^*$ and $q_2^{pl}=q^o$, and the transition probabilities are evaluated according to type i beliefs. Then, following the same steps as in our analysis of value functions in Appendix C.1, we obtain,

$$\begin{aligned}
\rho w_s^i(1) &= \beta_s^i q_s^o + \left(1-\beta_s^i\right) q_{s'}^o - q^*, \\
\text{where } \beta_s^i &= \frac{\rho + \lambda_{s'}^i}{\rho + \lambda_{s'}^i + \lambda_s^i}.
\end{aligned}$$

Here, β_s^i can be thought of as the expected discount time the investor spends in state s according to type i beliefs. We consider this equation for $s=2$ and take the derivative with respect to $\left(-\lambda_2^{o,pl}\right)$ to obtain,

$$\frac{\partial w_2^i(1)}{\partial\left(-\lambda_2^{o,pl}\right)} = \beta_2^i \frac{dq_2^o}{d\left(-\lambda_2^{o,pl}\right)} < 0.$$

Here, the inequality follows since reducing optimists' optimism reduces the price level in the common belief benchmark (see Section 4).

Note that the inequality, $\frac{\partial w_2^i(1)}{\partial\left(-\lambda_2^{o,pl}\right)} < 0$, holds for each belief type i. Using the continuity of the partial derivative function, $\frac{\partial w_2^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}$, we conclude that there exists $\overline{\alpha}$ such that $\left.\frac{\partial w_2^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda_2^{o,pl}=\lambda_2^o} < 0$ for each i, s and $\alpha \in (\overline{\alpha}, 1)$, completing the proof. $\square$

D. Appendix: Extension with investment and endogenous growth

Our baseline setup in the main text assumes there is no investment and the expected growth rate of capital is exogenous. In this appendix, we analyze a more general environment that relaxes these assumptions. We first present the environment, define the equilibrium, and provide a partial characterization. We then characterize this equilibrium when investors have common beliefs and generalize Proposition 1 to this setting.

D.1. Environment and equilibrium with investment

We focus on the components that are different than the baseline setting described in Section 3.

Potential output and endogenous growth. We modify the equation that describes the dynamics of capital (14) as follows,

$$\frac{dk_{t,s}}{k_{t,s}} = g_{t,s}dt + \sigma_s dZ_t \qquad \text{where } g_{t,s} \equiv \varphi\left(\iota_{t,s}\right) - \delta. \tag{D.1}$$

Here, $\iota_{t,s} = \frac{i_{t,s}}{k_{t,s}}$ denotes the investment rate, $\varphi\left(\iota_{t,s}\right)$ denotes a neoclassical production function for capital (we will work with a special case that will be described below), and δ denotes the depreciation rate. Hence, the growth of capital is no longer exogenous: it depends on the endogenous level of investment as well as depreciation.

Investment firms. To endogenize investment, we introduce a new set of firms, which we refer to as investment firms, that own and manage the aggregate capital stock. These firms rent capital to production firms to earn the instantaneous rental rate, $R_{t,s}$. They also make investment decisions to maximize the value of capital. Letting $\tilde{Q}_{t,s}$ denote the price of capital, the firm's investment problem can generally be written as,

$$\max_{\iota_{t,s}} \tilde{Q}_{t,s}\varphi\left(\iota_{t,s}\right) k_{t,s} - \iota_{t,s}k_{t,s}. \tag{D.2}$$

As before, we denote the price of the market portfolio per unit of capital with $Q_{t,s}$. In this case, the market portfolio represents a claim on investment firms as well as production firms. Hence, we have the inequality $\tilde{Q}_{t,s} \leq Q_{t,s}$, where the residual price, $Q_{t,s} - \tilde{Q}_{t,s}$, corresponds to the value of production firms per unit of capital. We make assumptions (that we describe below) so that output accrues to the investment firms in the form of return to capital, $y_{t,s} = R_{t,s}k_{t,s}$, and there are no monopoly profits. This in turn implies that the value of the market portfolio is equal to the value of capital (and the value of production firms is zero), that is,

$$Q_{t,s} = \tilde{Q}_{t,s}. \tag{D.3}$$

This simplifies the analysis by ensuring that we have only one price to characterize. Considering a different division of output between return to capital and profits will have a quantitative effect on investment, as illustrated by problem ($D.2$), but we conjecture that it would leave our qualitative results on investment unchanged. We leave a systematic exploration of this issue for further research.

Return of the market portfolio. The price of the market portfolio per unit of capital follows the same equation (15) as in the main text. The volatility of the market portfolio (absent state transitions) is also unchanged and given by σ_s. However, the return on the market portfolio conditional on no transition is slightly modified and given by,

$$r^m_{t,s} = \frac{y_{t,s} - \iota_{t,s} k_{t,s}}{Q_{t,s} k_{t,s}} + \left(g_{t,s} + \mu^Q_{t,s}\right). \tag{D.4}$$

Hence, the dividend yield is now net of the investment expenditures the (investment) firms undertake. In addition, the expected growth of the price of the market portfolio is now endogenous and given by $g_{t,s}$.

Nominal rigidities and equilibrium in goods markets. As before, the supply side of our model features nominal rigidities similar to the standard New Keynesian model that ensure output is determined by aggregate demand. In this case, demand comes from investment as well as consumption so we modify Eq. (19) as,

$$y_{t,s} = \eta_{t,s} A k_{t,s} = \int_I c^i_{t,s} di + k_{t,s}\iota_{t,s}, \text{ where } \eta_{t,s} \in [0,1]. \tag{D.5}$$

We also modify the microfoundations that we provide in Section B.1.2 so that all output accrues to investment firms as return to capital and there are no monopoly profits, that is,

$$R_{t,s} = A\eta_{t,s} \text{ and thus } y_{t,s} = R_{t,s} k_{t,s}. \tag{D.6}$$

We relegate a detailed description of these microfoundations to the end of this appendix.

Combining Eqs. (*D.5*) , (*D.4*), (22) and (17), we can also rewrite the instantaneous (expected) return to the market portfolio as,

$$r^m_{t,s} = \rho + g_{t,s} + \mu^Q_{t,s}.$$

Hence, as in the main text, the equilibrium dividend yield is equal to the consumption rate ρ.

The rest of the model is the same as in Section 3. We formally define the equilibrium as follows.

Definition 2. *The equilibrium with investment and endogenous growth is a collection of processes for allocations, prices, and returns such that capital evolves according to* (14)*, the price of market portfolio per capital evolves according to* (15)*, its instantaneous return (conditional on no transition) is given by* (*D.4*)*, investment firms maximize (cf. Eqs.* (*D.7*)*, investors maximize (cf. Appendix B.1.1), asset markets clear (cf. Eqs.* (17) *and* (18)*), production firms maximize (cf. Appendix D.3), goods markets clear (cf. Eq.* (19)*), all output accrues to agents in the form of return to capital* (*D.6*)*, the price of the market portfolio per unit of capital is the same as the price of capital (cf. Eq.* (*D.3*)*), and the interest rate policy follows the rule in* (21)*.*

We next provide a partial characterization of the equilibrium with investment.

Investors' optimality conditions. Eqs. (22 – 25) in the main text remain unchanged.

Investment firms' optimality conditions. Under standard regularity conditions for the capital production function, $\varphi(\iota)$, the solution to problem (*D.2*) is determined by the optimality condition,

$$\varphi'(\iota_{t,s}) = 1/Q_{t,s}.$$

We will work with the special and convenient case proposed by Brunnermeier and Sannikov (2016b): $\varphi(\iota) = \psi \log\left(\frac{\iota}{\psi}+1\right)$. In this case, we obtain the closed form solution,

$$\iota(Q_{t,s}) = \psi(Q_{t,s}-1). \tag{D.7}$$

The parameter, ψ, captures the sensitivity of investment to asset prices.

Growth-asset price relation. Note also that the amount of capital produced is given by,

$$\varphi(\iota(Q_{t,s})) = \psi q_{t,s}, \text{ where } q_{t,s} \equiv \log(Q_{t,s}). \tag{D.8}$$

The log price level, $q_{t,s}$, will simplify some of the expressions. Combining Eq. ($D.8$) with Eq. (14), we obtain Eq. (37) in the main text, which we replicate here for ease of exposition,

$$g_{t,s} = \psi q_{t,s} - \delta.$$

Hence, the expected growth rate of capital (and potential output) is now endogenous and depends on asset prices. Lower asset prices reduce investment, which translates into lower growth and lower potential output in future periods. As we will describe, this mechanism provides a new source of amplification.

Output-asset price relation. As in the main text, there is a tight relationship between output and asset prices as in the two period model. Specifically, Eq. (26) in the main text continues to apply and implies that aggregate consumption is a constant fraction of aggregate wealth. Plugging this into Eq. (19) and using the investment equation ($D.7$), we obtain Eq. (36) in the main text, which we replicate here for ease of exposition,

$$A\eta_{t,s} = \rho Q_{t,s} + \psi(Q_{t,s}-1) = (\rho+\psi) Q_{t,s} - \psi.$$

In this case, factor utilization (and output) depends on capital not only because consumption depends on asset prices through a wealth effect but also because investment depends on asset prices through a standard marginal-Q channel. Full factor utilization, $\eta_{t,s} = 1$, obtains only if the price of capital is at a particular level

$$Q^* \equiv \frac{A+\psi}{\rho+\psi}.$$

This is the efficient price level that ensures that the implied consumption and investment clear the goods market. Likewise, the economy features a demand recession, $\eta_{t,s} < 1$, if and only if the price of capital is strictly below Q^*.

Combining the output-asset price relation (together with $y_{t,s} = A\eta_{t,s}k_{t,s}$) with Eq. ($D.7$), we obtain $\frac{y_{t,s}-\iota_{t,s}k_{t,s}}{Q_{t,s}k_{t,s}} = \rho$. Using this expression along with Eq. (37), we can rewrite Eq. (16) as,

$$r^m_{t,s} = \rho + \psi q_{t,s} - \delta + \mu^Q_{t,s}. \tag{D.9}$$

Hence, a version of Eq. (28) in the main text continues to apply. In equilibrium, the dividend yield on the market portfolio is equal to the consumption rate ρ. Moreover, the growth rate of dividends is endogenous and is determined by the growth-asset price relation.

Combining the output-asset price relation with the interest rate policy in (21), we also summarize the goods market side of the economy with (29) as in the main text. In particular, the equilibrium at any time and state takes one of two forms. If the natural interest rate is nonnegative, then the interest rate policy ensures that the price per unit of capital is at the efficient level, $Q_{t,s} = Q^*$, capital is fully utilized, $\eta_{t,s} = 1$, and output is equal to its potential, $y_{t,s} = Ak_{t,s}$. Otherwise, the interest rate is constrained, $r^f_{t,s} = 0$, the price is at a lower level, $Q_{t,s} < Q^*$, and output is determined by aggregate demand according to Eq. (27).

As a benchmark, we characterize the first-best equilibrium without interest rate rigidities. In this case, there is no lower bound constraint on the interest rate, so the price of capital is at its efficient level at all times and states, $Q_{t,s} = Q^*$. Combining this with Eq. ($D.9$), we obtain $r^m_{t,s} = \rho + \psi q^* - \delta$, where $q^* = \log Q^*$. Substituting this into Eq. (23) and using Eq. (25), we solve for "rstar" as,

$$r^{f*}_s = \rho + \psi q^* - \delta - \sigma^2_s \text{ for each } s \in \{1, 2\}\,. \tag{D.10}$$

Hence, in the first-best equilibrium the risk premium shocks are fully absorbed by the interest rate. We next characterize the equilibrium with interest rate rigidities for the case in which investors have common beliefs.

D.2. Common beliefs Benchmark with Investment

Suppose investors have common beliefs (that is, $\lambda^i_s \equiv \lambda_s$ for each i). Substituting Eq. ($D.9$) into (23), we obtain the following analogue of the risk balance conditions (32),

$$\sigma_s = \frac{\rho + \psi q_s - \delta + \lambda_s \left(1 - \frac{Q_s}{Q_{s'}}\right) - r^f_s}{\sigma_s} \text{ for each } s \in \{1, 2\}\,. \tag{D.11}$$

The only difference is that the growth rate in each state is endogenous and described by the growth-asset price relation, $g_s \equiv \psi q_s - \delta$, where recall that $q_s = \log Q_s$ [cf. Eq. (37)]. We also make the following analogue of Assumption 1.

Assumption 1^{I}. $\sigma^2_2 > \rho + \psi q^* - \delta > \sigma^2_1$.

With this assumption, we conjecture that the low-risk-premium state 1 features positive interest rates, efficient asset prices, and full factor utilization, $r^f_1 > 0, q_1 = q^*$ and $\eta_1 = 1$, whereas the high-risk-premium state 2 features zero interest rates, lower asset prices, and imperfect factor utilization, $r^f_2 = 0, q_2 < q^*$ and $\eta_2 < 1$.

Equilibrium in the high-risk-premium state and amplification from the growth-asset price relation. Under our conjecture, the risk balance condition ($D.11$) for the high-risk state $s = 2$ can be written as,

$$\sigma_2 = \frac{\rho + \psi q_2 - \delta + \lambda_2 \left(1 - \frac{Q_2}{Q^*}\right)}{\sigma_2}. \tag{D.12}$$

As before, this equation illustrates an amplification mechanism: Since the recession reduces firms' earnings, a lower price level does not increase the dividend yield (captured by the constant dividend yield, $\rho = \frac{\rho Q_2}{Q_2}$). Unlike before, Eq. ($D.12$) illustrates a second amplification mechanism captured by the growth-asset price relation, $g_2 = \psi q_2 - \delta$. In particular, a lower price level lowers investment, which reduces the expected growth of potential output and profits, which in turn lowers the return to capital. The strength of this second mechanism depends on the sensitivity of investment to asset prices, captured by the term ψq_2. Figure 1 in the introduction presents a graphical illustration of the two amplification mechanisms.

The stabilizing force from price declines comes from the expected transition into the low-risk-premium state captured by the term, $\lambda_2 \left(1 - \frac{Q_2}{Q^*}\right)$. As before, to ensure that there exists an equilibrium with positive prices, we need a minimum degree of optimism, which we capture with the following analogue of Assumption 2.

Assumption 2^{I}. $\lambda_2 \geq \lambda_2^{\min}$, where $\lambda_2^{\min}$ is the unique solution to the following equation over the range $\lambda_2 \geq \psi$:

$$\rho + \psi q^* - \delta + \lambda_2^{\min} - \psi + \psi \log\left(\psi/\lambda_2^{\min}\right) = \sigma_2^2.$$

This assumption ensures that there exists a unique $Q_2 \in (0, Q^*)$ that solves Eq. ($D.12$) (see the proof at the end of this section).

Equilibrium in the low-risk-premium state. Under our conjecture, the risk balance condition ($D.11$) can be written as,

$$r_1^f = \rho + \psi q^* - \delta - \sigma_1^2 + \lambda_1 \left(1 - \frac{Q^*}{Q_2}\right) \tag{D.13}$$

As before, the interest rate adjusts to ensure that the risk balance condition is satisfied with the efficient price level, $Q_1 = Q^*$. For our conjectured equilibrium, we also assume an upper bound on λ_1 so that the implied interest rate is positive, $r_1^f > 0$, which we capture with the following analogue of Assumption 3.

Assumption 3^{I}. $\lambda_1 < \left(\rho + \psi q^* - \delta - \sigma_1^2\right) / \left(Q^*/Q_2 - 1\right)$, where $Q_2 \in (0, Q^*)$ solves Eq. (34).

As before, Eq. ($D.13$) implies that r_1^f is decreasing in the transition probability, λ_1, as well as in the asset price drop conditional on transition, Q^*/Q_2.

The following result summarizes the characterization of equilibrium and generalizes Proposition 1. The testable predictions regarding the effect of risk premium shocks on consumption, investment, and output follow by combining the characterization with Eqs. (26) , ($D.7$) , (36), and .

Proposition 5. *Consider the extended model with investment with two states, $s \in \{1, 2\}$, with common beliefs and Assumptions 1^I-3^I. The low-risk-premium state 1 features a positive interest rate, efficient asset prices and full factor utilization, $r_1^f > 0, Q_1 = Q^*$ and $\eta_1 = 1$. The high-risk state 2 features zero interest rate, lower asset prices, and a demand-driven recession, $r_2^f = 0, Q_2 < Q^*$, and $\eta_2 < 1$, as well as a lower level of consumption, $c_{t,2}/k_{t,2} = \rho Q_2$, investment, $i_{t,2}/k_{t,2} = \psi\left(Q_2 - 1\right)$, output, $y_{t,2}/k_{t,2} = \left(\rho + \psi\right) Q_2 - \psi$, and growth, $g_2 = \psi q_2 - \delta$. The price of capital in state 2 is characterized as the unique solution to Eq. ($D.12$), and the risk-free rate in state 1 is given by Eq. ($D.13$).*

Proof. Most of the proof is provided in the discussion leading to the proposition. The remaining step is to show that Assumptions 1^{I}-2^{I} ensure there exists a unique solution, $Q_2 \in (0, Q^*)$ (equivalently, $q_2 < q^*$) to Eq. ($D.12$).

To this end, we define the function,

$$f\left(q_2, \lambda_2\right) = \rho + \psi q_2 - \delta + \lambda_2 \left(1 - \frac{\exp\left(q_2\right)}{Q^*}\right) - \sigma_2^2.$$

The equilibrium price is the solution to, $f\left(q_2, \lambda_2\right) = 0$ (given λ_2). Note that $f\left(q_2, \lambda_2\right)$ is a concave function of q_2 with $\lim_{q_2 \to -\infty} f\left(q_2, \lambda_2\right) = \lim_{q_2 \to \infty} f\left(q_2, \lambda_2\right) = -\infty$. Its derivative is,

$$\frac{\partial f\left(q_2, \lambda_2\right)}{\partial q_2} = \psi - \lambda_2 \exp\left(q_2 - q^*\right).$$

Thus, for fixed λ_2, it is maximized at,

$$q_2^{\max}\left(\lambda_2\right) = q^* + \log\left(\psi/\lambda_2\right).$$

Moreover, the maximum value is given by

$$\begin{aligned} f\left(q_2^{\max}\left(\lambda_2\right),\lambda_2\right) &= \rho-\delta+\psi\left(q^*+\log\left(\psi/\lambda_2\right)\right)+\lambda_2\left(1-\exp\left(\log\left(\psi/\lambda_2\right)\right)\right)-\sigma_2^2 \\ &= \rho-\delta+\psi q^*+\psi\log\left(\psi/\lambda_2\right)+\lambda_2-\psi-\sigma_2^2. \end{aligned}$$

Next note that, by Assumption 1^{I}, the maximum value is strictly negative when $\lambda_2=\psi$, that is, $f\left(q_2^{\max}\left(\psi\right),\psi\right)<0$. Note also that $\frac{df\left(q_2^{\max}\left(\lambda_2\right),\lambda_2\right)}{d\lambda_2}=1-\frac{\psi}{\lambda_2}$, which implies that the maximum value is strictly increasing in the range $\lambda_2\geq\psi$. Since $\lim_{\lambda_2\to\infty} f\left(q_2^{\max}\left(\lambda_2\right),\lambda_2\right)=\infty$, there exists $\lambda_2^{\min}>\psi$ that ensures $f\left(q_2^{\max}\left(\lambda_2^{\min}\right),\lambda_2^{\min}\right)=0$. By Assumption 2^{I}, the transition probability satisfies $\lambda_2\geq\lambda_2^{\min}$, which implies that $f\left(q_2^{\max}\left(\lambda_2\right),\lambda_2\right)\geq 0$. By Assumption 1^{I}, we also have that $f\left(q^*,\lambda_2\right)<0$. It follows that, under Assumptions 1^{I}-2^{I}, there exists a unique price level, $q_2\in\left[q_2^{\max},q^*\right)$, that solves the equation, $f\left(q_2,\lambda_2\right)=0$. □

D.3. New Keynesian microfoundations for nominal rigidities with investment

In the rest of this appendix, we present the microfoundations for nominal rigidities that lead to Eqs. ($D.5$) and ($D.6$). The production structure is the same as in Appendix B.1.2. Specifically, there is a continuum of monopolistically competitive production firms that produce intermediate goods according to ($B.3$), and there is a competitive sector that produces the final good according to ($B.4$). This also implies the demand for production firms is given by($B.5$). One difference is that production firms do not own the capital but they rent it from investment firms at rate $R_{t,s}$. Hence, they choose how their capital input $k_{t,s}\left(\nu\right)$, in addition to their factor utilization rate, $\eta_{t,s}\left(\nu\right)$, as well as production and pricing decisions, $y_{t,s}\left(\nu\right),p_{t,s}\left(\nu\right)$.

These features ensure that the production firm's output will be split between their capital expenditures (that they pay to investment firms) and monopoly profits. To simplify the analysis, we make assumptions so that there are no monopoly profits in equilibrium (and all output accrues to investment firms as return to capital). Specifically, we assume the government taxes the firm's profits lump sum, and redistributes these profits to the firms in the form of a linear subsidy to capital.

Formally, we let $\Pi_{t,s}\left(\nu\right)$ denote the equilibrium pre-tax profits of firm ν (that will be characterized below). We assume each firm is subject to the lump-sum tax determined by the average profits of all firms,

$$T_{t,s}=\int_{\nu}\Pi_{t,s}\left(\nu\right)d\nu. \tag{D.14}$$

We also let $R_{t,s}-\tau_{t,s}$ denote the after-subsidy cost of renting capital, where $R_{t,s}$ denotes the equilibrium rental rate paid to investment firms, and $\tau_{t,s}$ denotes a linear subsidy paid by the government. We assume the magnitude of the subsidy is determined by the government's break-even condition,

$$\tau_{t,s}\int_{\nu}k_{t,s}\left(\nu\right)d\nu=T_{t,s}. \tag{D.15}$$

Without price rigidities, the firm chooses $p_{t,s}\left(\nu\right),k_{t,s}\left(\nu\right),\eta_{t,s}\left(\nu\right)\in\left[0,1\right],y_{t,s}\left(\nu\right)$, to maximize its (pre-tax) profits,

$$\Pi_{t,s}\left(\nu\right)\equiv p_{t,s}\left(\nu\right)y_{t,s}\left(\nu\right)-\left(R_{t,s}-\tau_{t,s}\right)k_{t,s}\left(\nu\right), \tag{D.16}$$

subject to the supply constraint in ($B.3$) and the demand constraint in ($B.5$). As in Appendix B.1.2, the demand constraint holds as equality. Then, the optimality conditions imply,

$$\eta_{t,s}(\nu) = 1 \text{ and } p_{t,s}(\nu) = \frac{\varepsilon}{\varepsilon - 1}\frac{R_{t,s} - \tau_{t,s}}{A}.$$

That is, the firm utilizes its capital at full capacity (as before) and it increases its capital input and production up to the point at which its price is a constant markup over its after-subsidy marginal cost. In a symmetric-price equilibrium, we further have, $p_{t,s}(\nu) = 1$. Using Eqs. ($B.3$) and ($D.15$), this further implies,

$$y_{t,s}(\nu) = y_{t,s} = Ak_{t,s} \text{ and } R_{t,s} = \frac{\varepsilon - 1}{\varepsilon}A + \tau_{t,s} = A. \tag{D.17}$$

That is, output is equal to potential output, and capital earns its marginal contribution to potential output (in view of the linear subsidies).

Now consider the alternative setting in which the firms have a preset nominal price that is equal across firms, $P_{t,s}(\nu) = P$. In particular, the relative price of a firm is fixed and equal to one, $p_{t,s}(\nu) = 1$. The firm chooses the remaining variables, $k_{t,s}(\nu), \eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu)$, to maximize its (pre-tax) profits, $\Pi_{t,s}(\nu)$, subject to the supply constraint in ($B.3$) and the demand constraint, ($B.5$). Combining the constraints and using $p_{t,s}(\nu) = 1$, the firm's problem can be written as,

$$\max_{\eta_{t,s}(\nu), k_{t,s}(\nu)} A\eta_{t,s}(\nu) k_{t,s}(\nu) - (R_{t,s} - \tau_{t,s}) k_{t,s}(\nu) \text{ s.t. } 0 \leq \eta_{t,s}(\nu) \leq 1 \text{ and } A\eta_{t,s}(\nu) k_{t,s}(\nu) \leq y_{t,s}.$$

We conjecture an equilibrium in which $R_{t,s} = \tau_{t,s}$ and firms choose symmetric capital inputs, $k_{t,s}(\nu) = k_{t,s}$. Under this equilibrium, the marginal cost of renting capital is zero, $R_{t,s} - \tau_{t,s} = 0$. This verifies that it is optimal for firms to choose symmetric inputs, $k_{t,s}(\nu) = k_{t,s}$. After substituting these expressions, the firm's problem becomes equivalent to its counterpart in Appendix B.1.2. Following the same steps there, the optimal factor utilization is given by $\eta_{t,s}(\nu) = \frac{y_{t,s}}{Ak_{t,s}} \leq 1$. Hence, output is determined by aggregate demand, $y_{t,s}$, subject to the capacity constraint, $\eta_{t,s}(\nu) \leq 1$.

In the conjectured equilibrium, the production firms choose the same level of inputs and factor utilization rates and produce the same level of output as each other. Therefore, they also have the same level of pre-tax profits. Using Eqs. ($D.16$) together with $R_{t,s} = \tau_{t,s} = 0$, we also calculate the pre-tax profit level as $\Pi_{t,s} = y_{t,s}$. Substituting this into Eqs. ($D.14$) and ($D.15$), we obtain $\tau_{t,s} = y_{t,s}/k_{t,s} = \eta_{t,s}A$. Substituting this into Eq. ($D.16$), we further obtain $R_{t,s} = y_{t,s}/k_{t,s} = \eta_{t,s}A$. This verifies the conjecture, $R_{t,s} = \tau_{t,s}$.

In sum, when the firms' nominal prices are fixed, aggregate output is determined by aggregate demand subject to the capacity constraint, which verifies Eq. ($D.5$). Moreover, thanks to lump-sum costs to profits and linear subsidies to capital, all output accrues to the investment firms as return to capital, which verifies Eq. ($D.6$).

E. Appendix: Data Details and Omitted Empirical Results

This appendix presents the details of the data sources and variable construction used in Section 7, and presents the empirical results (tables and figures) omitted from the main text.

House price index. We rely on the cross-country quarterly panel dataset described in Mack et al. (2011). The dataset is regularly updated and publicly available at https://www.dallasfed.org/institute/houseprice. We use the inflation-adjusted (real) house price index measure to construct the shock variable in our regression analysis (see (54)). Our country coverage is to a large extent determined by the availability of this measure, e.g., we exclude a few developed countries such as Portugal and Austria for which we do not have consistent data on real house prices.

Euro or Exchange Rate Mechanism (Euro/ERM) status. We hand-collect this data from various online sources. A country-quarter is included in the Euro/ERM sample if the country is a member of the Euro or the European Exchange Rate mechanism in most of the corresponding calendar year. Table 1 describes the Euro/ERM status by year for all countries in our sample.

GDP, consumption, investment. We obtain this data from the OECD's quarterly national accounts dataset (available at https://stats.oecd.org). We use the variables calculated according to the expenditure approach. The corresponding OECD subject codes are as follows:

- GDP: "B1_GE" (Gross domestic product – expenditure approach).
- Consumption: "P31S14_S15" (Private final consumption expenditure)
- Investment: "P51" (Gross fixed capital formation)

For each of these variables, we use the measures that are adjusted for inflation as well as seasonality. The OECD measure code is: "LNBQRSA" (National currency, chained volume estimates, national reference year, quarterly levels, seasonally adjusted).

Relative GDP (with PPP-adjusted prices in a common base year). We obtain an alternative GDP measure from the OECD's annual national accounts dataset (available at https://stats.oecd.org). We use the variable calculated according to the expenditure approach (with subject code "B1_GE"), measured with PPP-adjusted prices in a common base year. The OECD measure code is: "VPVOB" (Current prices, constant PPPs, OECD base year). We use the value of this measure in 1990 to weight all of our regressions (see (54)).

CPI. We obtain this data from the OECD's prices and purchasing power parities dataset (available at https://stats.oecd.org). We use the core CPI measure that excludes food and energy. The OECD subject code is: "CPGRLE" (Consumer prices - all items non-food, non-energy). We use the annual measure, which is less subject to seasonality, and we linearly interpolate this to obtain a quarterly measure.

Unemployment rate. We obtain this data from the OECD's key short-term economic indicators database (available at https://stats.oecd.org). We use the harmonized unemployment rate measure with seasonal adjustment and at quarterly frequency. The OECD subject code is "LRHUTTTT" (Harmonised unemployment rate: all persons, s.a).

The policy interest rate. Obtaining the policy interest rate is not as trivial as it might sound since different central banks conduct monetary policy in terms of different target rates (and sometimes without specifying a target rate, or by monitoring multiple rates). On the other hand, the selection does not substantially affect the results since short-term risk-free rates within a developed country are often highly correlated. Following Romer and Romer (2018), we use announced policy target rates when available, and otherwise we use collateralized short-term market rates (such as Repo rates or Lombard rates). For Eurozone countries, we use the local collateralized rate until the country joins the Euro, and we switch to the European Central Bank's (ECB) main refinancing operations (MRO) rate after the country joins the Euro.

For most of the countries, we construct our own measure of the policy interest rate according to the above selection criteria by using data from the Global Financial Data's GFDATABASE (GFD). This is a proprietary database that contains a wealth of information on various asset prices (see https://www.globalfinancialdata.com for details).

For a few countries (specified below), we instead rely on the Bank for International Settlements's (BIS) database on central bank policy interest rates (publicly available at https://www.bis.org/statistics/cbpol.htm). We switch to the BIS measure when we cannot construct an appropriate measure using the GFD; or when the BIS measure has greater coverage than ours and the two measures are highly correlated. From either database, we obtain monthly data and convert to quarterly data by averaging over the months within the quarter.

- United States: GFD ticker "IDUSAFFD" (USA Fed Funds Official Target Rate).
- United Kingdom: GFD ticker "IDGBRD" (Bank of England Base Lending Rate).
- Australia: GFD ticker "IDAUSD" (Australia Reserve Bank Overnight Cash Rate).
- South Korea: GFD ticker "IDKORM" (Bank of Korea Discount Rate).
- Germany: GFD ticker "IDDEULD" (Germany Bundesbank Lombard Rate) until the country joins the Euro. Afterwards, we use the ECB MRO rate. The corresponding GFD ticker is: "IDEURMW" (Europe Marginal Rate on Refinancing Operations).
- New Zealand: GFD ticker "IDNZLD" (New Zealand Reserve Bank Official Cash Rate).
- France: GFD ticker "IDFRARD" (Bank of France Repo Rate) until the country joins the Euro.
- Denmark: We use the BIS measure (highly correlated with our measure and greater coverage).
- Finland: GFD ticker "IDFINRM" (Bank of Finland Repo Rate) until the country joins the Euro.
- Sweden: GFD ticker "IDSWERD" (Sweden Riksbank Repo Rate).
- Israel: GFD ticker "IDISRD" (Bank of Israel Discount Rate).
- Italy: GFD ticker "IDITARM" (Bank of Italy Repo Rate) until the country joins the Euro.
- Spain: GFD ticker "IDESPRM" (Bank of Spain Repo Rate) until the country joins the Euro.
- Ireland: GFD ticker "IDIRLRD" (Bank of Ireland Repo Rate) until the country joins the Euro.
- Belgium: GFD ticker "IDBELRM" (Belgium National Bank Repo Rate) until the country joins the Euro.
- Greece: GFD ticker "IDGRC"D (Bank of Greece Discount Rate) until the country joins the Euro.

- Netherlands: GFD ticker “IDNLDRD” (Netherlands Bank Repo Rate) until the country joins the Euro.
- Norway: GFD ticker “IDNORRD” (Bank of Norway Sight Deposit Rate).
- Japan: GFD ticker “IDJPNCM” (Japan Target Call Rate). GFD data is missing from March 2001 until July 2006. BIS data is also missing for most of this period. We use other sources to hand-fill the interest rate over this period as being equal to 0% (see for instance, the data from St. Louis Fed at https://fred.stlouisfed.org/series/IRSTCI01JPM156N).
- Switzerland: We use the BIS measure (cannot identify an appropriate rate from the GFD).
- Canada: We use the BIS measure (highly correlated with our measure and greater coverage).

Stock prices. We obtain this data from the GFD. For each country, we try to pick the most popular stock price index (based on Internet searches). We obtain daily data and convert to quarterly data by averaging over all (trading) days within the quarter. We then divide this with our core CPI measure (see above) to obtain a real stock price series.

- United States: GFD ticker “_SPXD” (S&P500 Index)
- United Kingdom: GFD ticker “_FTSED” (UK FTSE100 Index).
- Australia: GFD ticker “_AXJOD” (Australia S&P/ASX 200 Index).
- South Korea: GFD ticker “_KS11D” (Korea SE Stock Price Index (KOSPI)).
- Germany: GFD ticker “_GDAXIPD” (Germany DAX Price Index).
- New Zealand: GFD ticker “_NZ15D” (NZSX-15 Index).
- France: GFD ticker “_FCHID” (Paris CAC-40 Index).
- Denmark: GFD ticker “_OMXC20D” (OMX Copenhagen-20 Index).
- Finland: GFD ticker “_OMXH25D” (OMX Helsinki-25 Index).
- Sweden: GFD ticker “_OMXS30D” (OMX Stockholm-30 Index).
- Israel: GFD ticker “_TA125D” (Tel Aviv SE 125 Broad Index).
- Italy: GFD ticker “_BCIJD” (Milan SE MIB-30 Index).
- Spain: GFD ticker “_IBEXD” (Madrid SE IBEX-35 Index).
- Ireland: GFD ticker “_ISEQD” (Ireland ISEQ Overall Price Index).
- Belgium: GFD ticker “_BFXD” (Belgium CBB Bel-20 Index).
- Greece: GFD ticker “_ATGD” (Athens SE General Index).
- Netherlands: GFD ticker “_AEXD” (Amsterdam AEX Stock Index).
- Norway: GFD ticker “_OSEAXD” (Oslo SE All-Share Index).
- Japan: GFD ticker “_N225D” (Nikkei 225 Stock Index).
- Switzerland: GFD ticker “_SSMID” (Swiss Market Index).

- Canada: GFD ticker "_GSPTSED" (Canada S&P/TSX 300 Index).

Earnings. We obtain monthly data on the price-earnings ratio of publicly traded firms from the GFD (typically constructed for a broad sample of stocks chosen by the GFD). We then combine this information with our nominal price index (using the price at the last trading day of the month) to construct a monthly series for earnings. We convert this to a quarterly measure by averaging over the months within the quarter. We then divide this by our core CPI measure to obtain a quarterly real earnings series for publicly traded firms.

GFD ticker for the price earnings ratio typically has the form "SY-three digit country code-PM" (e.g., the ticker for the United States is "SYUSAPM"). One exception is the United Kingdom for which the corresponding GFD code is "_PFTASD" (UK FT-Actuaries PE Ratio).

Credit expansion. Our measure of bank credit is based on Baron and Xiong (2017), who construct a variable, *credit expansion*, defined as the annualized past three-year change in bank credit to GDP ratio. Mathematically, it is expressed as

$$\text{credit expansion} = \frac{\Delta\left(\frac{\text{bank credit}}{\text{GDP}}\right)_t - \Delta\left(\frac{\text{bank credit}}{\text{GDP}}\right)_{t-12}}{12} \times 4, \tag{E.1}$$

where t denotes a quarter. Baron and Xiong (2017) construct this measure by merging data from two sources. Their main source is the "bank credit" measure from the BIS, which covers a large set of countries but is generally available only for postwar years. For this reason, Baron and Xiong (2017) also supplement it with the "bank loans" measure from Schularick and Taylor (2012), which covers fewer countries but more years. Since our panel starts in 1990, we ignore the second source and rely entirely on the BIS measure.

Specifically, we use the quarterly BIS database on credit to the nonfinancial sector (publicly available at https://www.bis.org/statistics/totcredit.htm). We obtain the measure "bank credit to the private nonfinancial sector" expressed in units of percentage of GDP (the corresponding BIS code is "Q:5A:P:B:M:770:A"), which enables us to construct the variable in ($E.1$). We verify that our variable is highly correlated with the measure constructed by Baron and Xiong (2017) (who generously shared their data with us)—the correlation coefficient for the available country-quarters is 0.975.

Following Baron and Xiong (2017), we also construct a "credit expansion-std" variable by standardizing the measure in ($E.1$) by its mean and standard deviation within each country. Since Baron and Xiong (2017) focus on predicting stock prices, they calculate the mean and the standard deviation using only past data so as to avoid any look-ahead bias. Since our focus is different, we ignore this subtlety and calculate the sample statistics using the entire data for the corresponding country (in the BIS database).

Table 1: Euro/ERM status by country and year

Country	1990	1991	1992	1993	1994	1995	1996	1997-2017
Belgium	1	1	1	1	1	1	1	1
Denmark	1	1	1	1	1	1	1	1
Finland	0	0	0	0	0	0	0	1
France	1	1	1	1	1	1	1	1
Germany	1	1	1	1	1	1	1	1
Greece	0	0	0	0	0	0	1	1
Ireland	1	1	1	1	1	1	1	1
Italy	1	1	1	0	0	0	0	1
Netherlands	1	1	1	1	1	1	1	1
Spain	1	1	1	1	1	1	1	1
Australia	0	0	0	0	0	0	0	0
Canada	0	0	0	0	0	0	0	0
Israel	0	0	0	0	0	0	0	0
Japan	0	0	0	0	0	0	0	0
Korea	0	0	0	0	0	0	0	0
NZL	0	0	0	0	0	0	0	0
Norway	0	0	0	0	0	0	0	0
Sweden	0	1	1	0	0	0	0	0
Switzerland	0	0	0	0	0	0	0	0
UK	0	1	1	0	0	0	0	0
USA	0	0	0	0	0	0	0	0

Euro status. Belgium, Finland, France, Germany, Ireland, Italy, Netherlands, Spain adopted the Euro in 1999. Greece adopted in 2001. Denmark hasn't adopted the Euro but is a member of the ERM.

Table 2: Summary statistics by ERM for the baseline regression sample

	ERM sample		Non-ERM sample		Difference	
	Mean	Std.Deviation	Mean	Std.Deviation	Mean	Std.Error
Δ log house prices (real)	0.0040	0.0183	0.0053	0.0181	-0.0013	(0.0023)
Δ log GDP (real)	0.0043	0.0128	0.0065	0.0093	-0.0022	(0.0010)
policy interest rate (nominal)	0.0232	0.0194	0.0352	0.0288	-0.0119	(0.0038)
Δ log CPI (core)	0.0041	0.0029	0.0046	0.0039	-0.0004	(0.0005)
Δ unemployment rate	-0.0000	0.0042	-0.0002	0.0030	0.0002	(0.0004)
Δ log investment (real)	0.0030	0.0535	0.0070	0.0297	-0.0040	(0.0021)
Δ log consumption (real)	0.0035	0.0103	0.0069	0.0095	-0.0034	(0.0008)
earnings to price ratio	0.0616	0.0409	0.0585	0.0227	0.0031	(0.0039)
Δ log stock prices (real)	0.0011	0.0974	0.0108	0.0820	-0.0097	(0.0047)
credit expansion	0.0175	0.0557	0.0136	0.0298	0.0040	(0.0079)
credit expansion-std	0.1715	1.2657	-0.0346	1.1128	0.2062	(0.1957)
Observations	821		1120		1941	

Δ represents quarterly change. Standard errors are Newey-West standard errors with a bandwidth of 20 quarters.

Table 3: Private housing wealth in 2005 (% of GDP) by Euro/ERM status

Country (Euro/ERM)	Housing wealth	Country (Non-Euro/ERM)	Housing wealth
Spain	414.33	Australia	301.32
Italy	271.25	USA	199.77
France	253.74	Korea	179.55
Netherlands	222.03	Japan	169.74
Germany	186.77	Canada	146.51
Denmark	168.45	Norway	139.48
		Sweden	132.10
Average	252.76	Average	181.21
GDP-weighted average	255.29	GDP-weighted average	191.64

Table 4: Stock market capitalization in 2005 (% of GDP) by Euro/ERM status

Country (Euro/ERM)	Market cap	Country (Non-Euro/ERM)	Market cap
Finland	102.48	Switzerland	229.68
Netherlands	87.37	Canada	129.84
Spain	82.95	UK	126.75
France	80.07	Australia	121.32
Belgium	74.47	Sweden	116.08
Denmark	67.30	USA	103.83
Greece	58.57	Korea	96.16
Ireland	53.90	Israel	86.04
Italy	43.08	Norway	79.94
Germany	42.01	Japan	61.89
Average	69.22	Average	115.16
GDP-weighted average	61.84	GDP-weighted average	120.26

Data sources. We obtain housing wealth to GDP ratio from the World Inequality Database (WID) which is publicly available at https://wid.world/. We construct the ratio by combining yearly series on "private housing assets" (WID indicator, "mpwhou") and "gross domestic product (WID indicator, "mgdpro").

We obtain stock market capitalization to GDP ratio as yearly series from the GFD. The corresponding ticker has the form "CM.MKT.LCAP.GD.ZS three digit country code" (e.g., the ticker for the United States is "CM.MKT.LCAP.GD.ZS USA").

For both tables, we construct the GDP-weighted averages by using our relative GDP measure (in 2005) described earlier in this appendix.

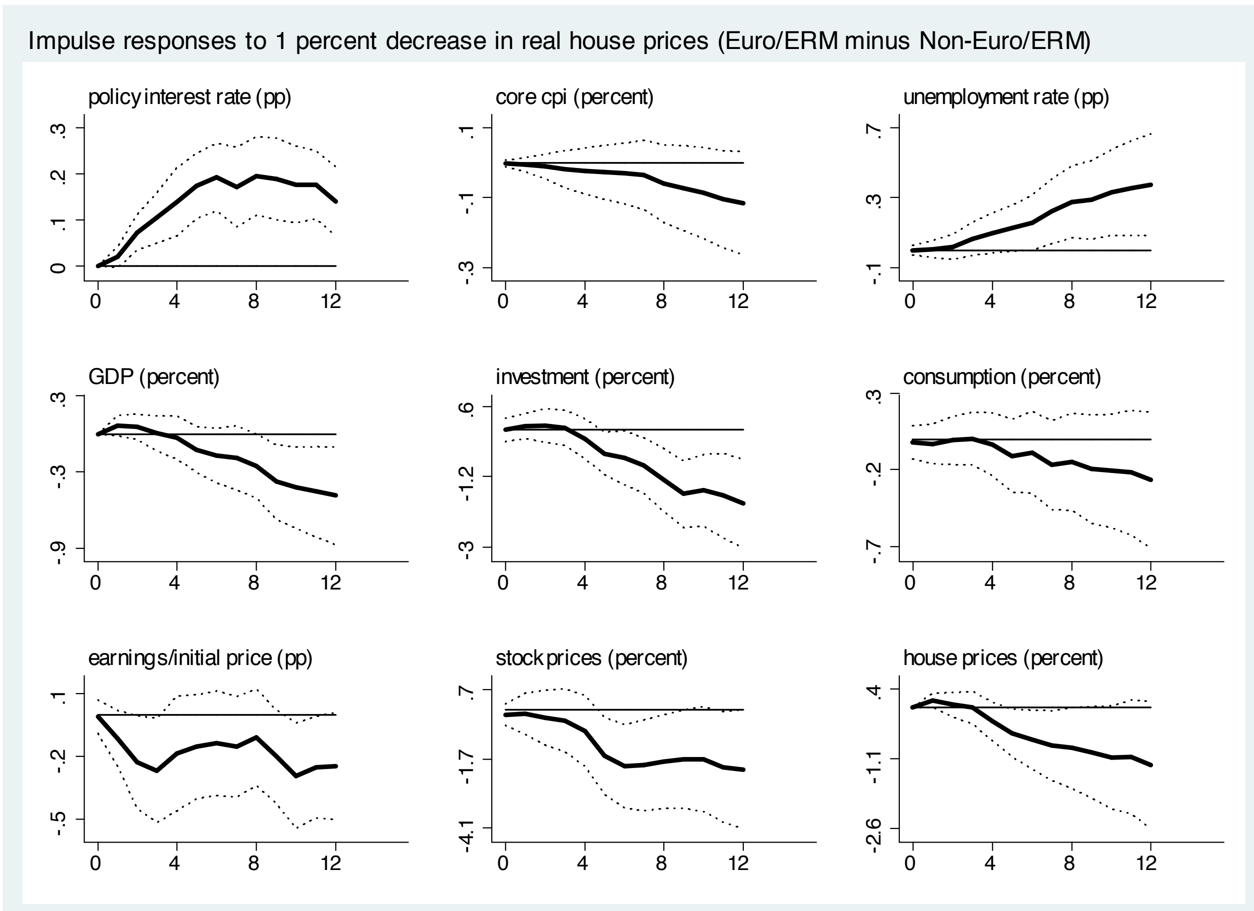

Figure 10: Differences in coefficients between the ERM and the non-ERM samples corresponding to the baseline regression results in Figure 6.

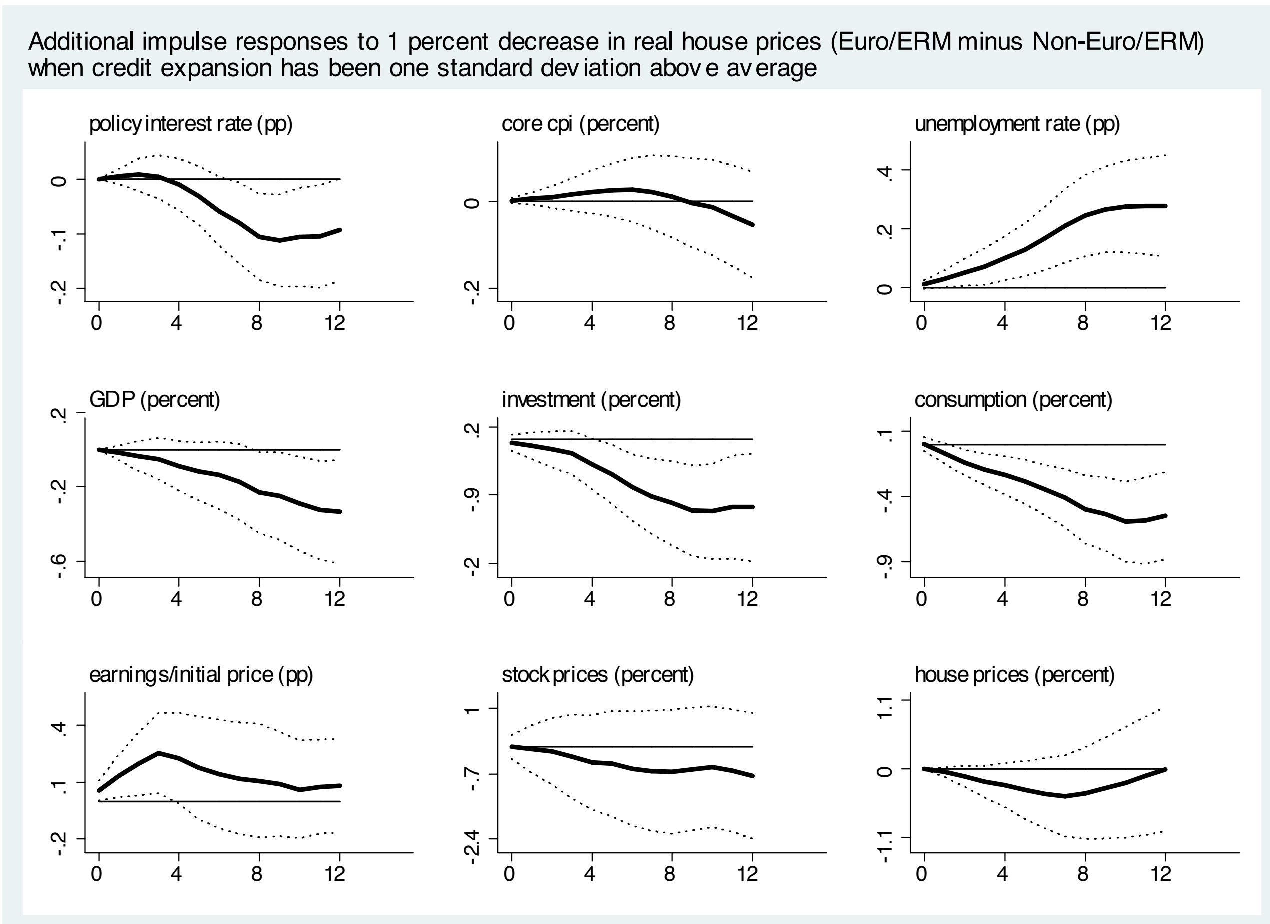

Figure 11: Differences in coefficients between the ERM and the non-ERM samples corresponding to the regression results with credit interaction in Figure 7.

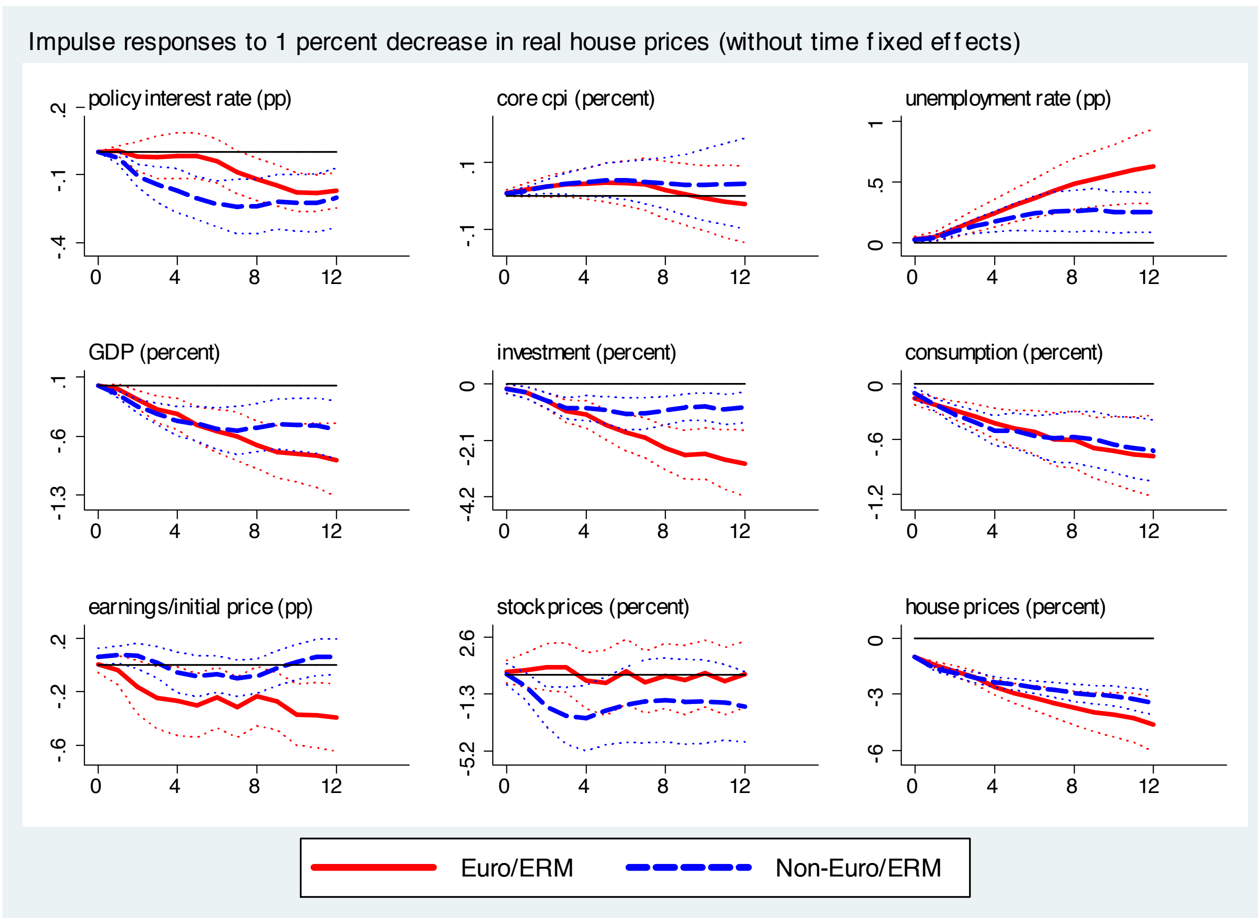

Figure 12: The analogues of the baseline regression results in Figure 6 with the difference that time fixed effects are excluded from the regressions.

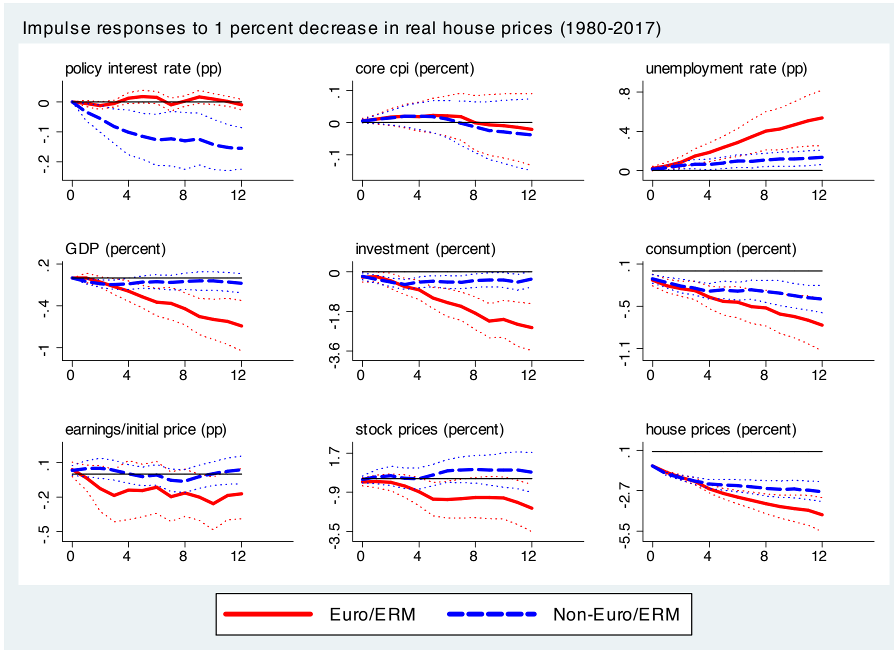

Figure 13: The analogues of the results in Figure 6 with a sample that starts in 1980Q1 (as opposed to 1990Q1).

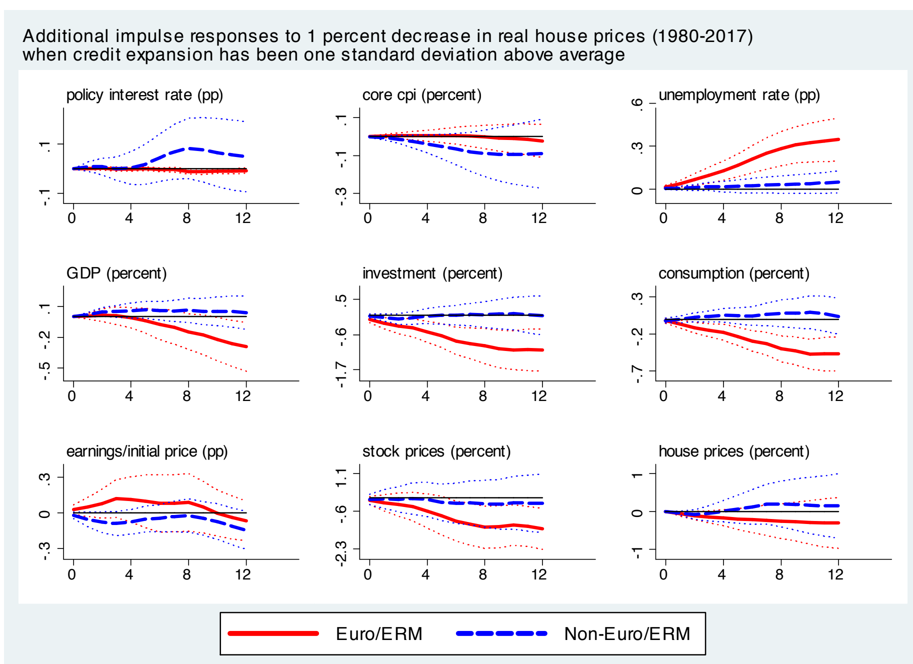

Figure 14: The analogues of the results in Figure 7 with a sample that starts in 1980Q1 (as opposed to 1990Q1).

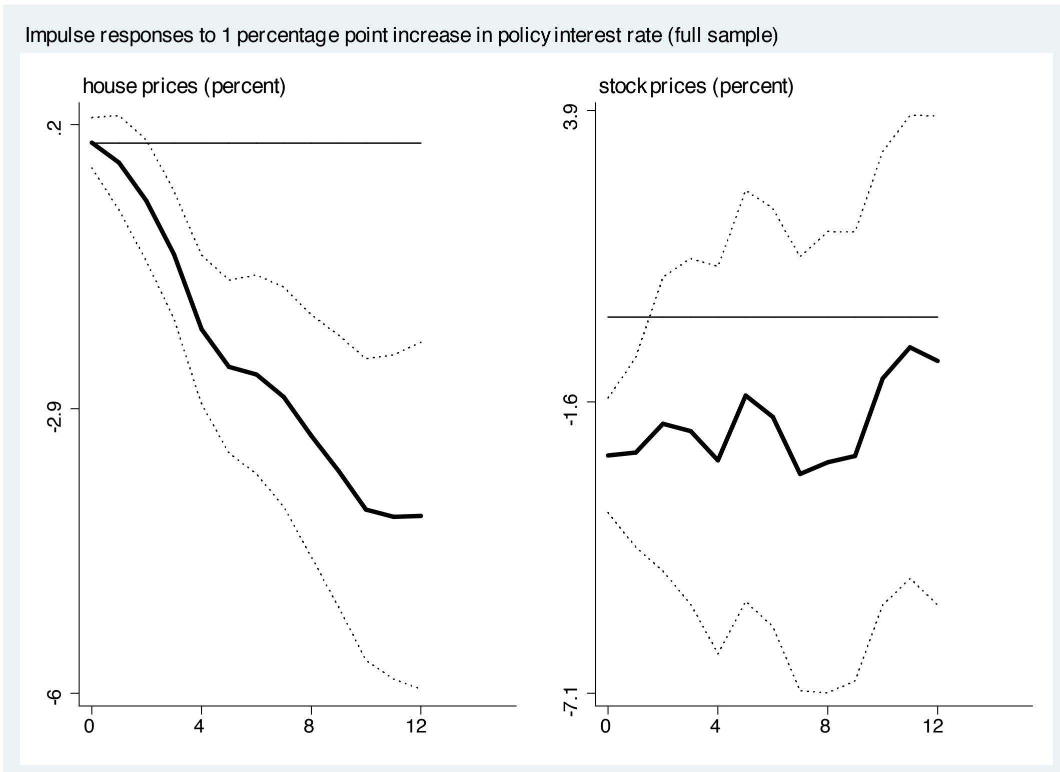

Figure 15: The analogues of the baseline regression results in Figure 6, where we consider shocks to the policy interest rate as opposed to house prices. Specifically, we run the analogue of the specification in (54) (on the full sample) where the shock variable is the level of the policy interest rate and the outcome variable is log house prices (left panel) or log stock prices (right panel). The solid lines plot the coefficients corresponding to the the policy interest rate variable. All regressions include time and country fixed effects; 12 lags of the level of the policy interest rate, contemporaneous value and 12 lags of the first difference of log GDP, 12 lags of the first difference of log house prices, and 12 lags of the first difference of log stock prices. The dotted lines show 95% confidence intervals calculated according to Newey-West standard errors with a bandwidth of 20 quarters. All regressions are weighted by countries' PPP-adjusted GDP in 1990. Data is unbalanced quarterly panel that spans 1990Q1-2017Q4. All variables except for the policy interest rate are adjusted for inflation. The sources and the definitions of variables are described earlier in this appendix.

Prudential Monetary Policy

Ricardo J. Caballero and Alp Simsek*

September 05, 2019

Abstract

Should monetary policy have a *prudential* dimension? That is, should policymakers raise interest rates to rein in financial excesses during a boom? We theoretically investigate this issue using an aggregate demand model with asset price booms and financial speculation. In our model, monetary policy affects financial stability through its impact on *asset prices*. Our main result shows that, when macroprudential policy is imperfect, small doses of *prudential monetary policy (PMP)* can provide financial stability benefits that are equivalent to tightening leverage limits. PMP reduces asset prices during the boom, which softens the asset price crash when the economy transitions into a recession. This mitigates the recession because higher asset prices support leveraged, high-valuation investors' balance sheets. An alternative intuition is that PMP raises the interest rate to create room for monetary policy to react to negative asset price shocks. The policy is most effective when there is extensive speculation and leverage limits are neither too tight nor too slack. When shadow banks are present, PMP can still replicate the benefits of macroprudential policy, but PMP is less effective (like macroprudential policy) because shadow banks respond by increasing their leverage.

JEL Codes: E00, E12, E21, E22, E30, E40, G00, G01, G11
Keywords: Speculation, leverage, aggregate demand, business cycle, effective lower bound, monetary policy, regulation, macroprudential policies, leaning against the wind, shadow banks.

Dynamic link to the most recent draft:
https://www.dropbox.com/s/nmzrbx964e12yus/PMP_public.pdf?dl=0

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1. Introduction

Should monetary policy have a *prudential* dimension? That is, should policymakers raise interest rates, or delay a cut, to rein in financial excesses during a boom? This question has occupied the minds of central bankers and monetary policy researchers for decades. At present, there are two dominant views. The *fully-separable* view contends that monetary policy should focus exclusively on its traditional mandate while delegating financial stability concerns to macroprudential policy (see, e.g., Weidmann (2018); Svensson (2018)). The *non-separable* view argues that, in practice, macroprudential policy might be insufficient to deal with financial excesses since its tools are limited and inflexible (see, e.g., Stein (2014); Gourio et al. (2018)). This debate has led to a growing literature investigating the costs and benefits of *prudential monetary policy (PMP)*. In this paper, we provide a new rationale for PMP, and we show that under appropriate circumstances it can be as effective as macroprudential policy. This equivalence is useful since, as highlighted by the non-separable view, monetary policy in practice is significantly nimbler than macroprudential policy when responding to cyclical fluctuations.[1]

PMP has obvious costs: it slows down the economy and leads to inefficient factor utilization during the boom. The benefits are less well understood. One of the main arguments for PMP is the *asset price channel:* monetary policy can mitigate the asset price boom and therefore make the subsequent crash smaller and less costly (see, e.g., Borio (2014); Adrian and Liang (2018)). This view is supported by evidence that monetary policy has a sizable, nearly immediate impact on asset prices. Despite its potential importance, there is little formal analysis on how the asset price channel of PMP works and whether (or when) it improves social welfare. We fill this gap by developing an aggregate demand model with asset price booms and speculation.

In our model, the economy transitions from a boom with high asset prices into a recession with low asset prices. The boom features *financial speculation*—investors with heterogeneous valuations trading risky financial assets amongst themselves. We focus on speculation among investors with heterogeneous beliefs (optimists and pessimists), but similar insights apply if speculation is driven by other forces such as heterogeneous risk tolerances (e.g., banks and households). The recession features *interest rate frictions*—factors that might constrain how the risk-free rate adjusts after a shock. We focus on the zero lower bound, but our mechanism applies for other constraints that limit downward adjustment in interest rates during recessions.

These ingredients make optimists' wealth share a key state variable for the economy. In particular, when optimists have more wealth in the recession, they push up asset prices and aggregate demand, softening the recession. However, individual optimists that take on leverage during the boom (and pessimists that lend to them) do not internalize the welfare effects of

[1] Adrian et al. (2017) summarize the results from a tabletop exercise conducted by the Federal Reserve that "aimed at confronting Federal Reserve Bank presidents with a plausible, albeit hypothetical, macro-financial scenario that would lend itself to macroprudential considerations...From among the various tools considered, tabletop participants found many of the prudential tools less attractive due to implementation lags and limited scope of application...Monetary policy came more quickly to the fore as a financial stability tool than might have been thought before the exercise."

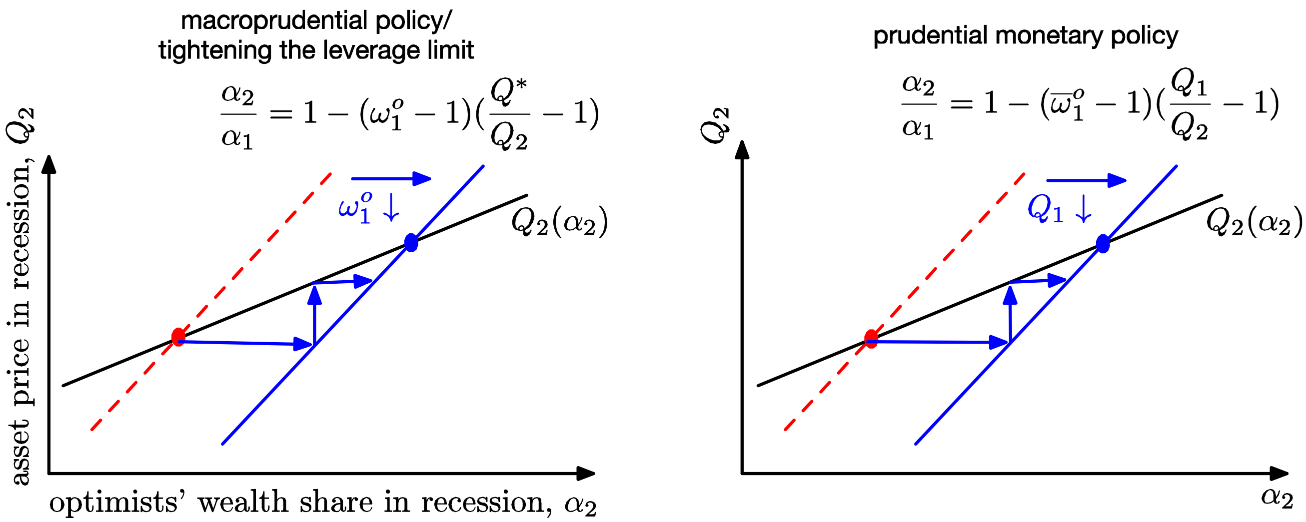

Figure 1: Graphical illustration of the relations that determine optimists' wealth share and the asset price in recession, (α_2, Q_2). The left (resp. right) panel illustrates the effect of macroprudential policy (resp. PMP).

optimists' wealth losses during the recession, which motivates policy interventions. Macroprudential policy is in theory the ideal tool for disciplining optimists' risk taking, but in practice it can be imperfect. Our main result shows that in such instances PMP can effectively reduce optimists' risk-exposure.

To illustrate this result, we introduce some notation and relations (we provide microfoundations in the main text). Specifically, let $s = 1$ and $s = 2$ denote the *boom* and the *recession* states, respectively. The economy is set in continuous time and transitions from the boom state to the recession state according to a Poisson process. Let α_s and Q_s denote optimists' wealth share and the price of capital (asset price) in state s, respectively. In the recession state $s = 2$, the price of capital is an increasing function of optimists' wealth share:

$$Q_2 = Q_2(\alpha_2). \tag{1}$$

In the boom state $s = 1$, optimists choose an above-average leverage ratio, $\omega_1^o > 1$. Therefore, if there is a transition to the recession state, their wealth share declines. Specifically, we have,

$$\frac{\alpha_2}{\alpha_1} = 1 - (\omega_1^o - 1)\left(\frac{Q_1}{Q_2} - 1\right), \tag{2}$$

where $Q_1/Q_2 > 1$ captures the magnitude of the price decline after the transition. Note that this equation also describes an increasing relation between optimists' wealth share, α_2, and the price of capital in the recession, Q_2 (since $\omega_1^o > 1$). Given a boom wealth share α_1, the equilibrium pair, (α_2, Q_2), corresponds to the intersection of two increasing relations (1) and (2), similar to Kiyotaki and Moore (1997). Figure 1 provides a graphical representation of these relations.

In this framework, aggregate demand is an increasing function of asset prices, so monetary policy can be described in terms of its effect on asset prices. As a benchmark, suppose the monetary authority sets interest rates in the boom to achieve asset prices and aggregate demand consistent with potential output, $Q_1 = Q^*$. In the recession, monetary policy is constrained, so asset prices and aggregate demand fall short of potential output, $Q_2 < Q^*$. A larger wealth share for optimists, α_2, increases asset prices and aggregate demand and softens the recession. This effect is an *aggregate demand externality,* which provides a rationale for prudential policies that improve optimists' wealth share in the recession, α_2.

Eq. (2) suggests that there are two prudential channels policymakers can use to increase α_2. First consider macroprudential policy that reduces optimists' leverage ratio, ω_1^o. This policy increases α_2 by reducing optimists' exposure to a given asset price decline, Q^*/Q_2. Second, suppose instead that optimists' leverage ratio is fixed, $\omega_1^o = \overline{\omega}_1^o$, due to either binding macroprudential policy or financial frictions, and consider PMP that reduces asset prices during the boom, $Q_1 < Q^*$. This policy increases α_2 by decreasing the size of the asset price decline, Q_1/Q_2, for a given level of optimists' exposure. Figure 1 shows that these two policies can achieve the same allocations, illustrating the logic behind our main result.

Moreover, as we shall see in the formal derivation, PMP lowers asset prices, $Q_1 < Q^*$, by setting the interest rate higher than the benchmark with conventional output stabilization ("rstar"). Thus, an equivalent intuition for our main result is that PMP raises the interest rate to create room for monetary policy to react to negative asset price shocks. This interpretation would not apply in the standard New Keynesian model where the severity of the recession depends only on the *level* of interest rates. In our model, the *path* of interest rates also matters because optimists' balance sheet is a key state variable that is affected by *changes* in asset prices.

PMP has two potential drawbacks relative to macroprudential policy. First, optimists' leverage ratio has to be constrained and must not react to a policy-induced change in asset prices, $\omega_1^o = \overline{\omega}_1^o$. In our model, when optimists are fully unconstrained, their leverage ratio adjusts to completely undo the prudential effects of monetary policy. That is, once ω_1^o adjusts, α_2 does not depend on Q_1. The intuition is that, since optimists perceive smaller risks after transition to a recession, they increase their leverage ratio. While this result is extreme and driven by specific features of our model (in particular, complete markets and no borrowing constraints), it provides a cautionary note and illustrates that PMP is more effective when there is *some* macroprudential policy that (imperfectly) restricts optimists' risk taking.

Second, even when monetary policy achieves the same prudential objectives as macroprudential policy, it is more costly because it lowers asset prices during the boom, $Q_1 < Q^*$, which reduces factor utilization below the efficient level. However, in a neighborhood of the price level that ensures efficient factor utilization (Q^*), these negative welfare effects are second order. On the other hand, the beneficial effects of softening the recession are first order. Our main result formalizes this insight and establishes that (when optimists are subject to some leverage limit) the first-order welfare effects of PMP are *exactly the same* as the effects of tightening the leverage limit directly. Put differently, *for small policy changes, PMP is as effective as macroprudential policy.* PMP increases unemployment in a booming economy, which has negligible costs, and reduces unemployment during a recession, which has sizeable benefits.

This discussion illustrates how our main result may apply beyond our particular model of recessions. For example, suppose the recession features no interest rate frictions, but there are *financial frictions* and *fire-sale prices* that increase in *experts'* wealth share. Suppose experts take on leverage during the boom to increase the size of their investments (as in Lorenzoni (2008)). The analogues of Eqs. (1) and (2) apply in this setting. Hence, as long as experts' leverage is constrained, PMP would improve experts' balance sheets in the recession and improve welfare. In this alternative setup, the policy would improve welfare by mitigating *fire-sale externalities*, whereas in our model PMP internalizes aggregate demand externalities.

We also characterize the optimal monetary policy in our environment and establish three comparative static results. First, the planner utilizes PMP more when leverage limits (or macroprudential policy) are at an intermediate level. Intuitively, when the limits are too loose, PMP is not worthwhile because it requires a large decline in Q_1 to push optimists against their constraints. Naturally, when the limits are already too tight, further tightening via PMP is not beneficial. These two extreme cases illustrate that macroprudential policy and PMP can be *complements* as well as *substitutes.* Second, as expected, the planner utilizes PMP more when she perceives a greater probability of transitioning into a recession. Finally, the planner utilizes PMP more when investors have greater disagreements about the risk of a recession. This result highlights that the policy is not driven by high asset prices *per se* (which is addressed by conventional monetary policy objectives) but by the financial speculation associated with episodes that concentrate risks on optimists' (or banks') balance sheets.

Finally, one of the main concerns in practice with respect to prudential policies is the presence of "shadow banks" (lightly regulated high-valuation agents). We extend our analysis to consider these agents and show that PMP can still replicate the financial stability benefits of macroprudential policy. However, both policies are weaker than when there are no shadow banks. The policies are weaker because of general equilibrium feedbacks: less regulated agents respond to the stabilizing benefits of either policy by increasing their leverage and risk taking.

Literature review. Our paper is part of a large literature that investigates the effect of monetary policy on financial stability. Adrian and Liang (2018) provide an excellent recent survey (see also Smets (2014)). As they note, easy monetary policy can generate financial vulnerabilities by fueling credit growth, exacerbating the maturity mismatch of financial intermediaries, and inflating asset prices. Our paper focuses on the asset-price channel, which is underexplored.

One strand of the literature emphasizes that loose monetary policy can reduce risk premia during the boom by exacerbating the "reach for yield" due to incentive problems or behavioral forces (see, e.g., Rajan (2006); Maddaloni and Peydró (2011); Borio and Zhu (2012); Morris and Shin (2014); Lian et al. (2018); Acharya and Naqvi (2018)). In our model, monetary policy does not directly affect the risk premium—it affects asset prices mainly through the traditional discount rate channel. Nonetheless, we find a role for PMP because the reduction in asset prices during the boom softens the asset price crash after transition to recession. Our channel is stronger (and it operates through the same key equations) if, as suggested by empirical evidence, monetary policy also affects the risk premium during the boom (e.g., Bernanke and Kuttner (2005); Hanson and Stein (2015); Gertler and Karadi (2015); Gilchrist et al. (2015)).

Our paper complements the literature emphasizing the credit channel. A number of papers show that monetary policy can affect financial stability by influencing credit growth or leverage. Woodford (2012) articulates this channel using a New Keynesian framework (that builds upon Curdia and Woodford (2010)) in which loose monetary policy increases the leverage of financial institutions (or borrowers), which in turn increases the probability of a crisis (by assumption). We show that monetary policy also affects the severity of future downturns by influencing asset prices during the boom. Moreover, our model does not require a financial crisis: there are benefits if the economy transitions into a plain-vanilla recession (in which monetary policy is constrained). Hence, our theoretical findings suggest that quantitative analyses that rely purely on the credit channel and financial crises likely underestimate the benefits of PMP.[2]

In our model, PMP causes an output gap during the boom, which generates a second-order welfare loss (for small changes in policy), and mitigates the output gap during the recession, which generates a first-order welfare gain. Kocherlakota (2014) and Stein (2014) derive similar insights by assuming that the Fed uses a quadratic loss function to penalize deviations of unemployment from its target. They show that targeting financial stability fits naturally into the Fed's dual mandate. Our model provides a microfoundation for their key assumption that accommodative monetary policy exacerbates financial vulnerability.

Our paper is part of a growing theoretical literature that analyzes the interactions between macroprudential and monetary policies in environments with aggregate demand externalities (see, e.g., Korinek and Simsek (2016); Farhi and Werning (2016); Rognlie et al. (2018)).[3] Most of these papers conclude that financial stability issues are best addressed with macroprudential policy. We depart from this literature by assuming that macroprudential policy can be constrained, and we find a role for monetary policy. We also investigate the asset price channel, whereas Korinek and Simsek (2016) and Farhi and Werning (2016) focus on credit. Rognlie et al. (2018) analyze investment and show that incorporating this ingredient would strengthen our main result. When alternative policies are imperfect, PMP can be used to reduce investment

[2] A growing empirical literature has documented that rapid credit growth is associated with more frequent and more severe financial crises (e.g., Borio and Drehmann (2009); Jordà et al. (2013)). Recent work uses the empirical estimates from this literature to calibrate Woodford-style models and quantify the costs and benefits of PMP. Svensson (2017); IMF (2015) argue that the costs of this policy exceed the benefits, whereas Gourio et al. (2018); Adrian and Liang (2018) find mixed effects.

[3] Several papers investigate the relationship between macroprudential and monetary policies but focus on other frictions, such as pecuniary externalities or moral hazard, e.g., Stein (2012); Collard et al. (2017); Martinez-Miera and Repullo (2019). A vast literature theoretically investigates macroprudential policy but doesn't focus on nominal rigidities or monetary policy (see, e.g., Dávila and Korinek (2017) and the references therein).

during the boom. PMP creates pent-up investment demand that raises investment, asset prices, and aggregate demand during the recession.

Finally, although our mechanism is more general, our specific model with belief disagreements and speculation is related to a large finance literature (e.g., Lintner (1969); Miller (1977); Harrison and Kreps (1978); Scheinkman and Xiong (2003); Fostel and Geanakoplos (2008); Geanakoplos (2010); Simsek (2013a,b); Iachan et al. (2015); Cao (2017); Heimer and Simsek (2018)). Similar to Caballero and Simsek (2017), we analyze speculation when aggregate demand can influence output due to interest rate rigidities. We depart from our earlier work by assuming that financial markets are incomplete due to exogenous leverage limits (see Remark 3). This assumption ensures that monetary policy affects the extent of speculation.

In Section 2 we introduce the basic environment, and provide a partial characterization of the equilibrium. In Section 3, we characterize the equilibrium in the recession state and illustrate the aggregate demand externalities that motivate policy interventions. In Section 4, we characterize the equilibrium in the boom state for a benchmark case without PMP, and illustrate how macroprudential policy can improve welfare. In Section 5, we introduce PMP and establish our main results regarding its (local) equivalence with macroprudential policy. In Section 6, we characterize the optimal PMP in our setting and establish its comparative statics. In Section 7, we add "shadow banks" to our framework and analysis. Section 8 concludes and is followed by several appendices that contain omitted derivations and proofs.

2. Environment and equilibrium

In this section we introduce our general dynamic environment. We then provide a definition and a partial characterization of the equilibrium. In subsequent sections we further analyze this equilibrium under different assumptions about monetary policy.

Potential output and risk premium shocks. The economy is set in infinite continuous time, $t \in [0, \infty)$, with a single consumption good and a single factor of production, capital. Let $k_{t,s}$ denote the capital stock at time t in the aggregate state $s \in S$.

The rate of capital utilization is endogenous and denoted by $\eta_{t,s} \in [0, 1]$. When utilized at this rate, $k_{t,s}$ units of capital produce

$$A\eta_{t,s}k_{t,s} \tag{3}$$

units of the consumption good. The capital stock follows the process

$$\frac{dk_{t,s}/dt}{k_{t,s}} = g_s - \delta\left(\eta_{t,s}\right). \tag{4}$$

The depreciation function $\delta\left(\eta_{t,s}\right)$ is increasing. Hence, Eqs. (3) and (4) illustrate that utilizing capital at a higher rate allows the economy to produce more current output at the cost of faster depreciation and slower output growth. Without nominal rigidities, there is an optimal level of capital utilization denoted by η^*, which we characterize in the subsequent analysis. With nominal rigidities, the economy may operate below this level of utilization, $\eta_{t,s} \leq \eta^*$, due to aggregate demand shortages.

Eq. (4) also illustrates that the expected growth rate of capital (before depreciation) is given by g_s, which is an exogenous parameter. The states, $s \in S$, differ only in terms of g_s. For simplicity, we assume there are three states, $s \in \{1, 2, 3\}$. The economy starts in state $s = 1$. While in states $s \in \{1, 2\}$, the economy transitions into state $s' \equiv s + 1$ according to a Poisson process that we describe below. Once the economy reaches $s = 3$, it stays there forever.

We assume the parameters satisfy $g_2 < \min(g_1, g_3)$. We envision a scenario in which the economy starts in the boom state with a relatively high growth rate, eventually enters a recession state with a low growth rate, then returns to an absorbing recovery state with a high growth rate. Accordingly, we refer to states $1, 2$, and 3 as "the boom," "the recession," and "the recovery," respectively. For analytical tractability, we focus on a single business cycle. Figure 2 illustrates the timeline of events for a particular realization of state transitions.

Remark 1 (Broadening the interpretation of expected growth fluctuations)**.** *We view the changes in the expected growth rate, g_s, as a device to capture more broadly "time-varying risk premia": that is, fluctuations in risky asset prices that are unrelated to short-run fundamentals (i.e., the current supply-determined output level). In Caballero and Simsek (2017), we formalize this intuition by showing that (in a two period model) changes in g_s generate the same effect on asset prices and economic activity as changes in risk or risk aversion. A large literature documents that time-varying risk premia are a pervasive phenomenon in financial markets (see Cochrane (2011); Campbell (2014) for recent reviews).*

Transition probabilities and belief disagreements. We let $\lambda_s^i > 0$ denote investor i's belief about the Poisson transition probability from state s into state $s' = s + 1$. Since state $s = 3$ is an absorbing state, we have $\lambda_3^i = 0$ for each i. For the remaining states, we assume there are two types of investors, $i \in \{o, p\}$. Type o investors are "optimists," and type p investors are "pessimists." We denote the difference between perceived transition probabilities for optimists and pessimists by

$$\Delta\lambda_s = \lambda_s^o - \lambda_s^p.$$

We assume the belief differences satisfy:

Assumption 1. $\Delta\lambda_1 < 0$ and $\Delta\lambda_2 > 0$.

When the economy is in the boom state $s = 1$, optimists assign a smaller transition probability to the recession state $s = 2$. When the economy is in the recession state, they assign a greater transition probability to the recovery state $s = 3$.

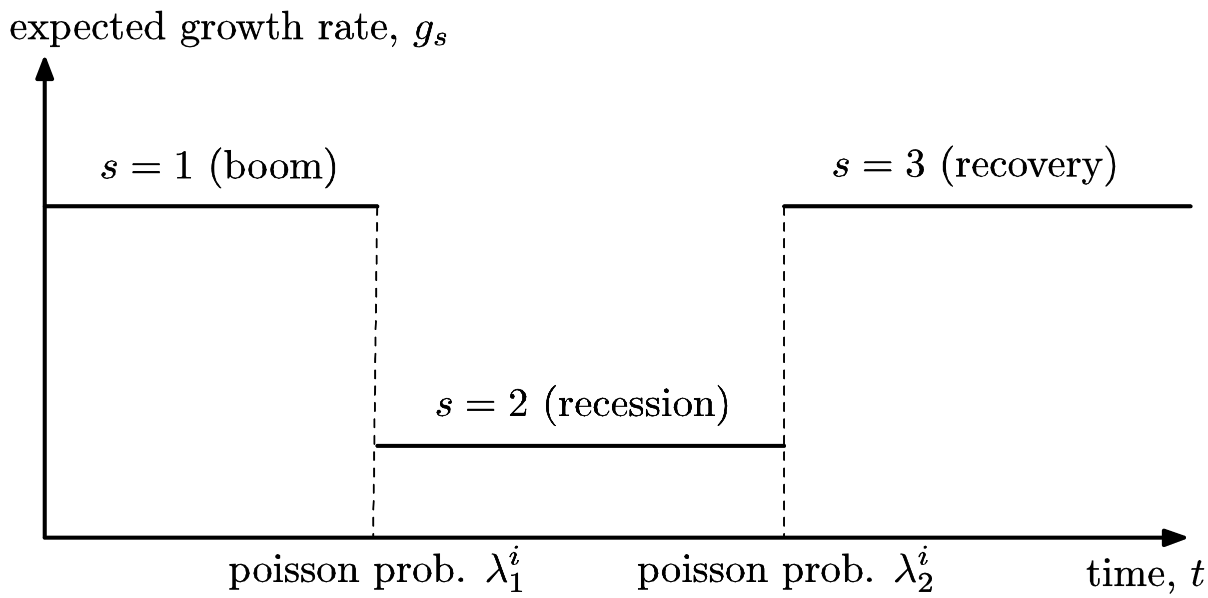

Figure 2: The timeline of events.

Remark 2 (Broadening the interpretation of disagreements)**.** *We view disagreements about transition probabilities as a convenient modeling device to capture heterogeneous asset valuations. The key aspects of "optimists" is that they value risky assets more than "pessimists," so that: (i) during the boom, they take on leverage, and (ii) during the recession, they increase risky asset prices. These aspects would be the same with other modeling devices such as heterogeneous risk aversion or Knightian uncertainty. Consequently, we can also think of "optimists" as banks (or institutional investors) that are more risk tolerant and less Knightian than households ("pessimists").*

Menu of financial assets. There are two types of financial assets. First, there is a market portfolio that represents a claim on all output (which accrues to production firms as earnings). We let $Q_{t,s}k_{t,s}$ denote the price of the market portfolio, so $Q_{t,s}$ is the price per unit of capital. We let $r_{t,s}$ denote the instantaneous expected return on the market portfolio conditional on no transition. Second, there is a risk-free asset in zero net supply. We denote its instantaneous return by $r^f_{t,s}$.

In Caballero and Simsek (2017), we allow for Arrow-Debreu securities that enable investors to trade the transition risk. In this paper, we assume financial markets are incomplete and thus investors speculate by adjusting their position on the market portfolio, i.e., changing their leverage ratio (see also Remark 3).

Market portfolio price and return. Absent state transitions, the price of capital $Q_{t,s}$ follows an endogenous, deterministic path. Using Eq. (4), the growth rate of the price of the market portfolio is given by

$$\frac{d\left(Q_{t,s}k_{t,s}\right)/dt}{Q_{t,s}k_{t,s}} = g_s - \delta\left(\eta_{t,s}\right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}},$$

where we use the notation $\dot{X} \equiv dX/dt$. Consequently, the return of the market portfolio absent state transitions can be written as

$$r_{t,s} = \frac{y_{t,s}}{Q_{t,s}k_{t,s}} + g_s - \delta\left(\eta_{t,s}\right) + \frac{\dot{Q}_{t,s}}{Q_{t,s}}. \tag{5}$$

Here, $y_{t,s}$ denotes the endogenous level of output at time t. Therefore, the first term captures the "dividend yield" component of return. The second term captures the capital gain conditional on no transition, which reflects the expected growth of capital and its price.

Portfolio choice. Investors are identical except for their beliefs about state transitions, λ_s^i. They continuously make consumption and portfolio allocation decisions. Specifically, at any time t and state s, investor i has some financial wealth denoted by $a_{t,s}^i$. She chooses her consumption rate, $c_{t,s}^i$, and the fraction of her wealth to allocate to the market portfolio, $\omega_{t,s}^i$. The residual fraction, $1-\omega_{t,s}^i$, is invested in the risk-free asset.

Note that $\omega_{t,s}^i$ also captures the investors' *leverage ratio*. We impose a leverage limit in the boom state $s=1$:

$$\omega_{t,1}^i \leq \overline{\omega}_{t,1}, \tag{6}$$

where we require $\overline{\omega}_{t,1} \geq 1$ (to ensure market clearing). We allow for $\overline{\omega}_{t,1} = \infty$, in which case the leverage limit never binds. Our main result applies when the leverage limit may bind. This constraint can capture a government-imposed leverage limit. It can also capture a market-imposed leverage limit due to unmodeled financial frictions such as moral hazard, adverse selection, lenders' uncertainty or their desire for safety. In fact, we can flexibly accommodate these frictions because we allow the leverage limit to change as the boom persists and risk conditions evolve. For simplicity, we assume that the leverage limit applies only in the boom state—adding this constraint to the recession (or recovery) states does not affect our qualitative findings.

For analytical tractability, we assume investors have log utility. The investors' problem (at time t and state s) can then be written as

$$V_{t,s}^i\left(a_{t,s}^i\right) = \max_{\left[c_{\tilde{t},\tilde{s}},\omega_{\tilde{t},\tilde{s}}^i\right]_{\tilde{t}\geq t,\tilde{s}}} E_{t,s}^i\left[\int_t^\infty e^{-\rho\tilde{t}}\log c_{\tilde{t},\tilde{s}}^i d\tilde{t}\right] \tag{7}$$

$$\text{s.t.} \begin{cases} da_{t,s}^i = \left(a_{t,s}^i\left(r_{t,s}^f + \omega_{t,s}^i\left(r_{t,s} - r_{t,s}^f\right)\right) - c_{t,s}\right)dt & \text{absent transition,} \\ a_{t,s'}^i = a_{t,s}^i\left(1 + \omega_{t,s}^i \frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right) & \text{if there is a transition to state } s' \neq s \end{cases} \tag{8}$$

and $\omega_{t,1}^i \leq \overline{\omega}_{t,1}$.

Here, $E^i_{t,s}[\cdot]$ denotes the expectation operator corresponding to investor i's beliefs for state transition probabilities.

Equilibrium in asset markets. Asset markets clear when the total wealth held by investors is equal to the value of the market portfolio both before and after the portfolio allocation decisions:

$$a^o_{t,s} + a^p_{t,s} = \omega^o_{t,s} a^o_{t,s} + \ \omega^p_{t,s} a^p_{t,s} = Q_{t,s} k_{t,s}. \tag{9}$$

When the conditions in (9) are satisfied, the market clearing condition for the risk-free asset (which is in zero net supply) holds.

Nominal rigidities and equilibrium in goods markets. The supply side of our model features nominal rigidities similar to the New Keynesian model. There is a continuum of competitive production firms that own the capital stock and produce the final good. For simplicity, these production firms have pre-set nominal prices that never change. Firms choose their capital utilization rate, $\eta_{t,s}$, to maximize their market value subject to demand constraints. They take into account that greater $\eta_{t,s}$ increases production according to Eq. (3) and that it leads to faster capital depreciation according to Eq. (4).

First consider the benchmark case without price rigidities. In this case, firms solve the problem:

$$\max_{\eta_{t,s}} \eta_{t,s} A k_{t,s} - \delta\left(\eta_{t,s}\right) Q_{t,s} k_{t,s}. \tag{10}$$

The optimality condition is given by

$$\delta'\left(\eta_{t,s}\right) Q_{t,s} = A. \tag{11}$$

That is, the frictionless level of utilization ensures that the marginal depreciation rate is equal to the marginal product of capital.

Next consider the case with price rigidities. In this case, firms solve problem (10) with the additional constraint that their output is determined by aggregate demand. As in the New Keynesian model, firms optimally meet this demand as long as their price exceeds their marginal cost. In a symmetric environment, the real price per unit of consumption good is one for all firms, and each firm's marginal cost is given by $\frac{\delta'(\eta_{t,s})Q_{t,s}}{A}$. Therefore, firms' optimality condition can be written as[4]

$$y_{t,s} = \eta_{t,s} A k_{t,s} = c^o_{t,s} + c^p_{t,s} \qquad \text{as long as } \delta'\left(\eta_{t,s}\right) Q_{t,s} \leq A. \tag{12}$$

Moreover, all output accrues to production firms in the form of earnings. Hence, the market portfolio can be thought of as a claim on all production firms.

[4] If instead the marginal cost exceeded the price, $\frac{\delta Q_{t,s}}{A} > 1$, then these firms would choose $\eta_{t,s} = 0$ and produce $y_{t,s} = 0$. This case does not emerge in equilibrium.

Interest rate rigidity and monetary policy. Our assumption that production firms do not change their prices implies that the aggregate nominal price level is fixed. The real risk-free interest rate, then, is equal to the nominal risk-free interest rate, which is determined by the monetary authority's interest rate policy. We assume there is a lower bound on the nominal interest rate, which we set as zero for convenience: $r^f_{t,s} \geq 0$.

We model monetary policy as a sequence of interest rates, $\left\{r^f_{t,s}\right\}_{t,s}$, and implied levels of factor utilization and asset price levels, $\left\{\eta_{t,s}, Q_{t,s}\right\}_{t,s}$, chosen subject to the zero lower bound constraint. Absent price rigidities, factor utilization and asset price levels satisfy condition (11). Therefore, we define the *conventional output-stabilization policy* as

$$r^f_{t,s} = \max\left(0, r^{f*}_{t,s}\right) \text{ for each } s, \tag{13}$$

where $r^{f*}_{t,s}$ ("rstar") is recursively defined as the instantaneous interest rate that obtains when condition (11) holds and the planner follows the output-stabilization policy in (13) at all future times and states.

Our goal is to understand whether the planner might want to use monetary policy for prudential purposes in the boom state. In particular, we assume the planner follows the conventional output-stabilization policy in (13) for the recession and the recovery states $s \in \{2, 3\}$, but she might deviate from this rule in the boom state $s = 1$. For now, we allow the planner to choose an arbitrary path, $\left\{r^f_{t,1}, Q_{t,1}, \eta_{t,1}\right\}_t$, that is consistent with the equilibrium conditions. We specify the monetary policy further in Section 5 and define the equilibrium below.

Definition 1. *The equilibrium is a collection of processes for allocations, prices, and returns such that capital evolves according to Eq. (4), its instantaneous return is given by Eq. (5), investors maximize their expected utility subject to a leverage limit in the boom state (cf. problem 7), asset markets clear (cf. Eq. (9)), goods markets clear (cf. Eq. (12)), and the monetary authority follows the conventional output-stabilization policy in states $s \in \{2, 3\}$ [cf. Eq. (13)] and chooses a feasible path $\left\{r^f_{t,1}, Q_{t,1}, \eta_{t,1}\right\}_t$ in state $s = 1$.*

We next provide a generally applicable partial characterization of the equilibrium. In subsequent sections, we use this characterization to describe the equilibrium in the different states and policy regimes.

2.1. Equilibrium in the goods market

We start by establishing the equilibrium conditions in the goods market. In view of log utility, the investor's consumption is a constant fraction of her wealth, regardless of her portfolio choice:

$$c^i_{t,s} = \rho a^i_{t,s}. \tag{14}$$

This leads to a tight relationship between output and asset prices. Combining Eqs. (14) and (9) implies that aggregate consumption is a constant fraction of aggregate wealth,

$$c_{t,s}^{o}+c_{t,s}^{p}=\rho Q_{t,s}k_{t,s}.$$

Combining this result with the goods market clearing condition in Eq. (12), we obtain *the output-asset price relation*,

$$A\eta_{t,s}=\rho Q_{t,s}. \tag{15}$$

Intuitively, greater asset prices increase aggregate demand, output, and factor utilization. Combining Eqs. (11) and (15), we find that the efficient level of output utilization solves

$$\delta'\left(\eta^{*}\right)\eta^{*}=\rho. \tag{16}$$

Note that optimal capital utilization is the same across all states. We assume the following regularity conditions on the depreciation function to ensure that there exists a unique solution to Eq. (16):

Assumption 2. $\delta\left(\eta\right)$ is strictly increasing and convex over $\mathbb{R}_{+}$ with $\delta'\left(0\right)<\rho$ and $\lim_{\eta\to\infty}\delta'\left(\eta\right)\geq\rho$.

Combining Eqs. (15) and (16), we find that there is an efficient asset price level:

$$Q^{*}=\frac{A\eta^{*}}{\rho}. \tag{17}$$

This is the level of asset prices such that the associated aggregate demand leads to efficient capital utilization (and ensures that actual output is exactly at potential output). When $Q_{t,s}<Q^{*}$, we have $\eta_{t,s}<\eta^{*}$: capital is utilized below its efficient level, which we interpret as a demand recession. Note also that, using the one-to-one relationship between factor utilization and asset prices in (15), we have $\frac{\eta_{t,s}}{\eta^{*}}=\frac{Q_{t,s}}{Q^{*}}$: the degree of underutilization relative to the efficient level is proportional to the ratio of the asset price level to the efficient asset price level.

Next note that we can use Eqs. (12) and (15) to rewrite Eq. (5) as

$$r_{t,s}=\rho+g_{s}-\delta\left(\frac{Q_{t,s}}{Q^{*}}\eta^{*}\right)+\frac{\dot{Q}_{t,s}}{Q_{t,s}}. \tag{18}$$

In equilibrium, the dividend yield on the market portfolio is equal to the consumption rate ρ.

2.2. Equilibrium in asset markets

We next establish the equilibrium conditions in asset markets. For these markets, the key state variable is investors' relative wealth shares, which we define as

$$\alpha_{t,s}^{i} \equiv \frac{a_{t,s}^{i}}{Q_{t,s}k_{t,s}} \text{ for } i \in \{o, p\}. \tag{19}$$

Note that investors' wealth shares sum to one, $\alpha_{t,s}^{o} + \alpha_{t,s}^{p} = 1$ [cf. Eq. (9)].

In the appendix, we characterize investors' wealth share after a transition in terms of their leverage ratio

$$\frac{\alpha_{t,s'}^{i}}{\alpha_{t,s}^{i}} - 1 = \left(\omega_{t,s}^{i} - 1\right) \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}. \tag{20}$$

When the transition increases the asset price, $Q_{t,s'} > Q_{t,s}$, an investor's wealth share increases after the transition, $\alpha_{t,s'}^{i} > \alpha_{t,s}^{i}$, if and only if she has above-average leverage, $\omega_{t,s}^{i} > 1$. The converse happens if the transition decreases the asset price.

Note also that Eq. (20) establishes a one-to-one relationship between $\alpha_{t,s'}^{i}$ and $\omega_{t,s}^{i}$ (as long as $Q_{t,s'} \neq Q_{t,s}$, which is the case in our model). Hence, we can think of the investor as choosing her wealth share after transition, $\alpha_{t,s'}^{i}$, and adjusting her leverage ratio to obtain this outcome. Thus, we can state the investor's portfolio optimality condition as

$$r_{t,s} - r_{t,s}^{f} + \lambda_{s}^{i} \frac{\alpha_{t,s}^{i}}{\alpha_{t,s'}^{i}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} \geq 0, \tag{21}$$

with equality when the leverage limit doesn't bind (see Appendix A.1 for a derivation). As long as the investor is unconstrained, she invests in the market portfolio until the risk-adjusted expected excess return is zero. The risk-adjusted return captures aggregate price changes $\left(\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}}\right)$ as well as the adjustment of marginal utility relative to other investors if there is a transition $\left(\frac{\alpha_{t,s}^{i}}{\alpha_{t,s'}^{i}}\right)$. For the equilibria we analyze, the leverage limit never binds for pessimists. Consequently, the optimality condition (21) always holds as equality for pessimists but it might apply as inequality for optimists.

Finally, combining Eqs. (9), (19) and (20), we can see that asset markets clear as long as investors' wealth shares after transition, $\left\{\alpha_{t,s'}^{i}\right\}_{i\in\{o,p\}}$, sum to one. Therefore, the equilibrium in asset markets reduces to finding wealth shares that solve (21) for each type and that satisfy $\alpha_{t,s'}^{o} + \alpha_{t,s'}^{p} = 1$.

Next consider the evolution of investors' wealth shares if there is no state transition. In Appendix A.1.2, we show that

$$\frac{\dot{\alpha}_{t,s}^{i}}{\alpha_{t,s}^{i}} = \lambda_{s}^{p} \frac{\alpha_{t,s}^{p}}{\alpha_{t,s'}^{p}} \left(1 - \frac{\alpha_{t,s'}^{i}}{\alpha_{t,s}^{i}}\right). \tag{22}$$

Pessimists' beliefs (superscript p) appear in this expression because the optimality condition (21) always holds as equality for them, so we can use their beliefs to price assets. This expression illustrates that investors face a trade-off across states. If an investor chooses $\alpha^i_{t,s'} > \alpha^i_{t,s}$ (resp. $\alpha^i_{t,s'} < \alpha^i_{t,s}$) so that her wealth share increases (resp. decreases) after a state transition, then she also has $\dot{\alpha}^i_{t,s} < 0$ (resp. $\dot{\alpha}^i_{t,s} > 0$) so her wealth share shrinks (resp. grows) if there is no state transition.

Special case with non-binding leverage limits. When the leverage limit doesn't bind for optimists, these equations can be simplified further. In particular, Eq. (21) holds as equality for both types of investors, which implies $\lambda^o_s \frac{\alpha^o_{t,s}}{\alpha^o_{t,s'}} = \lambda^p_s \frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}}$. Combining this equality with the market clearing condition (9), we obtain a closed-form solution:

$$\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}} = \frac{\lambda^i_s}{\overline{\lambda}_{t,s}} \text{ where } \overline{\lambda}_{t,s} = \alpha_{t,s}\lambda^o_{t,s} + (1-\alpha_{t,s})\,\lambda^o_{t,s}. \tag{23}$$

Here $\overline{\lambda}_{t,s}$ denotes the wealth-weighted average of the transition probability. After substituting this expression into Eq. (22), we solve for investors' wealth dynamics as:

$$\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} = -\left(\lambda^i_s - \overline{\lambda}_{t,s}\right). \tag{24}$$

These expressions are intuitive. When type i investors assign an above-average probability to transition, $\lambda^i_s > \overline{\lambda}_{t,s}$, their wealth share increases after a transition but drifts downward absent a transition. Conversely, when investors assign a below-average transition probability, their wealth share declines after a transition but drifts upward absent a transition.

Remark 3 (Role of market incompleteness due to binding leverage limits)**.** *Eqs. (21 – 24) clarify the difference of this model with the one in Caballero and Simsek (2017). Specifically, Eqs. (23) and (24) are the same as their counterparts in Caballero and Simsek (2017), where we allow investors to trade transition risks via Arrow-Debreu securities. The intuition is that, as long as the leverage limit does not bind, the market portfolio and the risk-free asset are sufficient to dynamically complete the market. The main difference in this setting is that the leverage limit* can *bind, in which case the wealth-share dynamics are different than in Caballero and Simsek (2017) and are characterized by Eqs. (21) and (22).*

3. The recession and aggregate demand externalities

We next characterize the equilibrium in the recession state (as well as in the recovery state). We also illustrate the aggregate demand externalities that motivate policy intervention. Since our focus is on the boom state, we relegate the details to Appendix A.2 and state the key equations and the results relevant for our analysis. For the rest of the paper, with a slight abuse of notation, we often drop the superscript o from optimists' wealth share:

$$\alpha_{t,s} \equiv \alpha_{t,s}^{o}.$$

Pessimists' wealth share is the complement of this expression, $\alpha_{t,s}^{p} = 1 - \alpha_{t,s}$. We will describe the remaining equilibrium variables as functions of optimists' wealth share, so this convention will considerably simplify the notation.

Under appropriate parametric restrictions (Assumption A1) we show that the recovery state $s = 3$ features positive interest rates, efficient asset prices, and efficient factor utilization, $r_{t,3}^{f} > 0, Q_{t,3} = Q^*, \eta_{t,3} = \eta^*$, whereas the recession state $s = 2$ features zero interest rates, inefficiently low asset prices, and inefficient factor utilization, $r_{t,2}^{f} = 0, Q_{t,2} < Q^*, \eta_{t,2} < \eta^*$. The equilibrium in the recovery state is straightforward since there is no further transition and no speculation. We then proceed backwards, starting with a description of the equilibrium in the recession state.

Equilibrium in the recession. Since there is no leverage limit in this state, Eq. (21) holds as equality for both types of investors. We aggregate this expression across investors (using Eq. (23)), and substitute for $r_{t,2}$ from Eq. (18) and $Q_{t,3} = Q^*$, to obtain:

$$\rho + g_2 - \delta\left(\frac{Q_{t,2}}{Q^*}\eta^*\right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_{t,2}}{Q^*}\right) = r_{t,2}^{f}. \tag{25}$$

We refer to this expression as the *risk balance condition*: it says that the equilibrium risk-adjusted return on the market portfolio (evaluated with the wealth-weighted average belief) is equal to the risk-free interest rate.

As a preliminary step, consider the outcomes that would obtain if the interest rate were unconstrained. In this case, substituting $Q_{t,2} = Q^*$ into the risk balance condition (25), we obtain an expression for the output-stabilizing interest rate: $r_{t,2}^{f,*} = \rho + g_2 - \delta(\eta^*)$. For intuition, consider the effect of lowering g_2. This exerts downward pressure on asset prices due to low expected growth in output and earnings. Monetary policy responds by lowering the risk-free interest rate, $r_{t,2}^{f,*}$, and keeps asset prices at the efficient level, $Q_{t,2} = Q^*$. By lowering the risk-free rate, monetary policy ensures that investors continue to hold the market portfolio at the efficient asset price level, even though they expect low output growth.

We assume g_2 is sufficiently low so that the implied output-stabilizing interest rate violates the lower bound, $r_{t,2}^{f,*} < 0$. Consider the outcomes with a binding interest rate lower bound. Substituting $r_{t,2}^{f} = 0$ into the risk balance condition (25), we obtain an expression that characterizes the asset price, $Q_{t,2}$. Intuitively, the only way condition (25) can be satisfied when $r_{t,2}^{f}$ cannot decline below zero is for $Q_{t,2}$ to fall below below Q^*. This asset price decline increases the return of the market portfolio, which in turn ensures that investors continue to hold the market portfolio despite lower expected output growth. However, the decline in $Q_{t,2}$ also lowers aggregate spending and triggers a demand recession.

Importantly, Eq. (25) suggests that, when the wealth-weighted belief is more optimistic (greater $\overline{\lambda}_{t,2}$), a smaller decline in $Q_{t,2}$ is sufficient to reestablish the risk balance condition. We verify this intuition in the appendix. Formally, we characterize the asset price and optimists' wealth share, $(Q_{t,2}, \alpha_{t,2})$, as the solution to a differential equation in the time domain (see Eq. (A.9)). We write the solution as $Q_{t,2} = Q_2(\alpha_{t,2})$ for each $\alpha_{t,2} \in [0,1]$. We show that the function, $Q_2(\cdot)$, satisfies

$$Q_2(\alpha) < Q^* \text{ and } \frac{dQ_2(\alpha)}{d\alpha} > 0 \text{ for each } \alpha \in (0,1). \tag{26}$$

In particular, a greater wealth-share for optimists increases the asset price and brings it closer to the frictionless level.

Recall from Eq. (15) that there is a one-to-one relationship between asset prices and factor utilization. Hence, Eq. (26) implies $\eta_{t,2} < \eta^*$: the recession features an inefficiently low level of capital utilization. We capture the welfare costs of underutilization with the concept of a *gap value function,* which we first introduced in Caballero and Simsek (2017).

Gap value function. To define the gap value function, let b denote a superscript representing beliefs about transition probabilities. The planner can have different beliefs from optimists and pessimists, so b takes one of three values $\{o, p, pl\}$. For a fixed b, we use $V_{t,s}^{i,b}\left(a_{t,s}^i\right)$ to denote type i investors' equilibrium value calculated according to type b beliefs. In view of log utility, the value function takes the form

$$V_{t,s}^{i,b}\left(a_{t,s}^i\right) = \frac{\log\left(a_{t,s}^i / Q_{t,s}\right)}{\rho} + v_{t,s}^{i,b}.$$

The normalized value function $v_{t,s}^{i,b}$ captures the value when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^i = Q_{t,s}$). We further decompose this term as follows:

$$v_{t,s}^{i,b} = v_{t,s}^{i*,b} + w_{t,s}^{b}. \tag{27}$$

The *frictionless value function* $v_{t,s}^{i*,b}$ is the value that obtains in a counterfactual economy where the evolution of wealth shares are left unchanged but asset prices are equal to the frictionless level, $Q_{t,s} = Q^*$ for each t, s. This captures all determinants of welfare (including the benefits/costs from speculation) except for suboptimal factor utilization. The residual term, $w_{t,s}^b$, corresponds to the *gap value function*. This term captures the welfare losses due to suboptimal factor utilization evaluated according to investors' preferences (and type b beliefs).

In the appendix, we formalize this intuition by establishing that the gap value function solves the following differential equation:

$$\begin{aligned}\rho w_{t,s}^{b}-\frac{\partial w_{t,s}^{b}}{\partial t} &= W\left(Q_{t,s}\right)+\lambda_{s}^{b}\left(w_{t,s'}^{b}-w_{t,s}^{b}\right), \\ \text{where } W\left(Q_{t,s}\right) &\equiv \log\frac{Q_{t,s}}{Q^{*}}-\frac{1}{\rho}\left(\delta\left(\frac{Q_{t,s}}{Q^{*}}\eta^{*}\right)-\delta\left(\eta^{*}\right)\right).\end{aligned} \tag{28}$$

The function $W(Q_{t,s})$ is strictly concave with a maximum at $Q_{t,s}=Q^{*}$ and maximum value equal to zero, $W(Q^{*})=0$ (cf. Eq. (16)). $W(Q_{t,s})\leq 0$ captures the instantaneous losses in welfare when the asset price (and therefore factor utilization) deviates from its efficient level, $Q_{t,s}\neq Q^{*}$. Therefore, the gap value $w_{t,s}^{b}$ corresponds to the present discounted value of expected welfare losses due to price rigidities and inefficient factor utilization.

In our welfare analysis, we mostly focus on the gap value function calculated according to the planner's belief, $b=pl$. This sidesteps questions about whether speculation increases or reduces welfare (see Brunnermeier et al. (2014) for further discussion). Our analysis aligns with the mandates of monetary policy in practice: the planner in our model exclusively focuses on minimizing output gaps relative to a frictionless benchmark (similar to Kocherlakota (2014) and Stein (2014)).[5] Following Brunnermeier et al. (2014), we assume the planner's beliefs are in the convex hull of optimists' and pessimists' beliefs: $\lambda_{1}^{pl}\in[\lambda_{1}^{o},\lambda_{1}^{p}]$ and $\lambda_{2}^{pl}\in[\lambda_{2}^{p},\lambda_{2}^{o}]$. Our results are qualitatively robust to the choice of planner's beliefs in these sets.

Gap value in the recession: Aggregate demand externalities. In the appendix, we show that the planner's gap value function in the recession can be written as $w_{t,2}^{pl}=w_{2}^{pl}(\alpha_{t,2})$, where $w_{2}^{pl}(\cdot)$ is a function that satisfies:

$$w_{2}^{pl}(\alpha)<0 \text{ and } \frac{dw_{2}^{pl}(\alpha)}{d\alpha}>0 \text{ for each } \alpha\in(0,1). \tag{29}$$

As expected, the gap value is strictly negative. Moreover, a greater wealth-share for optimists shrinks the gap value. The welfare gap is smaller when optimists have more wealth, since optimists' wealth increases asset prices and aggregate demand and mitigates the underutilization of capital [cf. Eqs. (26) and (28)].

Note that optimists' wealth share is an *endogenous* and *aggregate* state variable that depends on the amount of financial speculation that takes place in the boom state. In particular, the positive relationship between optimists' wealth and the gap value in (29) illustrates *aggregate demand externalities* that motivate policy interventions during the boom. Individual optimists that take on leverage during the boom (and pessimists that lend to them) do not internalize the effects of their financial decisions on asset prices in the recession. In subsequent sections, we

[5] In Caballero and Simsek (2017), we illustrate that (under appropriate parametric restrictions) macroprudential policy that restricts investors' risk taking can generate a *Pareto* improvement in welfare. That is, the planner can make everyone better off even if she focuses on the *total value* (not just the gap value) and evaluates each investor's expected value *according to her own belief*.

investigate whether prudential policies can help correct these externalities.

4. The boom: benchmark without prudential monetary policy

We now turn to our main focus: the equilibrium in the boom state. In this section, we analyze the benchmark case without PMP, that is, when monetary policy follows the conventional output-stabilization policy in (13) in state $s = 1$. We use this setup to illustrate that macroprudential policy that tightens the leverage limit can internalize the aggregate demand externalities. In the next section we introduce PMP and show that it can accomplish similar financial stability objectives to macroprudential policy.

Recall that investors face a (possibly time-varying) leverage limit, $\omega_{t,1}^i \leq \overline{\omega}_{t,1}$. We assume the leverage limit can be written as a function of optimists' wealth share, $\overline{\omega}_{t,1} = \overline{\omega}_1(\alpha_{t,1})$. This assumption ensures that $\alpha_{t,1}$ is the only state variable. We denote the equilibrium variables as functions of optimists' wealth share and the leverage limit *function*: $\alpha_{t,2} = \alpha_2(\alpha, \overline{\omega}_1(\cdot))$ denotes optimists' wealth share *after* transition when their current wealth share is $\alpha_{t,1} = \alpha$ and the leverage limit is described by $\overline{\omega}_{t,1} = \overline{\omega}_1(\alpha_{t,1})$ for each t. We use the notation $\alpha_2(\alpha, \infty)$ to denote the equilibrium when there is no leverage limit: $\overline{\omega}_1(\alpha) = \infty$ for each α.

Under appropriate parametric restrictions (Assumptions A2-A3 in the appendix) we show that the boom without PMP features positive interest rates, efficient asset prices, and efficient factor utilization, $r_{t,1}^f > 0, Q_{t,1} = Q^*, \eta_{t,1} = \eta^*$. To characterize this equilibrium, consider the intermediate cases, $\alpha_{t,1} \in (0,1)$ (the corner cases are straightforward and relegated to the appendix). The leverage limit doesn't bind for pessimists but it might bind for optimists. Using Eq. (21) for pessimists, and substituting $r_{t,1}$ from Eq. (18) and $Q_{t,1} = Q^*, Q_{t,2} = Q_2(\alpha_{t,2})$, we obtain

$$r_1^f(\alpha, \overline{\omega}_1) = \rho + g_1 - \delta(\eta^*) - \lambda_1^p \frac{1-\alpha}{1-\alpha_2(\alpha, \overline{\omega}_1)} \left(\frac{Q^*}{Q_2(\alpha_2(\alpha, \overline{\omega}_1))} - 1 \right). \tag{30}$$

This is the risk balance condition according to pessimists (cf. Eq. (25)). The condition characterizes the output-stabilizing interest rate given investors' wealth shares. Assumption A2 ensures that $r_1^f(\alpha, \overline{\omega}_1) > 0$ when $\alpha = 0$, that is, the interest rate is above the lower bound if pessimists dominate.

Hence, it remains to characterize the function $\alpha_2(\alpha, \overline{\omega}_1)$. First consider the special case without a leverage limit, $\overline{\omega}_1 = \infty$ for each α. In this case, Eq. (23) provides a closed-form solution:

$$\alpha_2(\alpha, \infty) = \alpha \frac{\lambda_1^o}{\overline{\lambda}_1(\alpha)} < \alpha. \tag{31}$$

Recall that we use the notation $\alpha_2(\alpha, \infty)$ to denote optimists' equilibrium wealth share without a leverage limit. The expression $\overline{\lambda}_1(\alpha) \equiv \alpha\lambda_1^o + (1-\alpha)\lambda_1^p$ denotes the wealth-weighted average probability as a function of optimists' wealth share. Using Eq. (20), we can solve for the corresponding leverage ratio in closed form:

$$\omega_1^o(\alpha,\infty) = 1 + \frac{1-\frac{\lambda_1^o}{\overline{\lambda}_1(\alpha)}}{\frac{Q^*}{Q_2\left(\frac{\alpha\lambda_1^o}{\overline{\lambda}_1(\alpha)}\right)}-1} > 1. \tag{32}$$

Optimists have above-average leverage during the boom, which induces a decline in their wealth share after transition to the recession.

Next consider the case with a leverage limit. Suppose $\overline{\omega}_1(\alpha) \leq \omega_1^o(\alpha,\infty)$ so that the limit binds (the other case is the same as before). Then, optimists' leverage ratio is determined by the limit:

$$\omega_1^o(\alpha,\overline{\omega}_1) = \overline{\omega}_1(\alpha). \tag{33}$$

To find optimists' wealth share after transition, we consider Eq. (20) for the boom state $s = 1$:

$$\begin{aligned} \frac{\alpha_2(\alpha,\overline{\omega}_1)}{\alpha} &= 1-(\overline{\omega}_1(\alpha)-1)\left[\frac{Q_1}{Q_2}-1\right], \\ \text{where } Q_1 &= Q^* \text{ and } Q_2 = Q_2(\alpha_2(\alpha,\overline{\omega}_1)). \end{aligned} \tag{34}$$

The first line of this expression is the microfounded version of Eq. (2) from the introduction. The second line substitutes the equilibrium prices for the boom and the recession states. The last equation is the microfounded version of Eq. (1). As illustrated by Figure 1, the equilibrium can be visualized as the intersection of two increasing relations. In Appendix A.3, we show that under appropriate regularity conditions (Assumption A3), Eq. (34) has a unique solution that satisfies $\alpha_2(\alpha,\overline{\omega}_1) \in [\alpha_2(\alpha,\infty),\alpha]$.

Finally, applying Eq. (22), we obtain the dynamics of optimists' wealth share absent a transition as

$$\frac{\dot{\alpha}_{t,1}}{\alpha_{t,1}} = \lambda_1^p \frac{1-\alpha_{t,1}}{1-\alpha_2(\alpha_{t,1},\overline{\omega}_1)}\left(1-\frac{\alpha_2(\alpha_{t,1},\overline{\omega}_1)}{\alpha_{t,1}}\right) \leq (1-\alpha_{t,1})(\lambda_1^p-\lambda_1^o). \tag{35}$$

The weak inequality is satisfied as equality when the leverage limit doesn't bind (i.e., when α_2 is given by Eq. (31)). It is also easy to see that $\dot{\alpha}_{t,1}/\alpha_{t,1}$ is a decreasing function of α_2: if optimists obtain a greater wealth share after transition to recession, then their wealth share grows more slowly if there is no transition.

To summarize the equilibrium without PMP, the asset price during the boom is at its efficient level, $Q_1(\alpha,\overline{\omega}_1) = Q^*$, and the equilibrium interest rate is given by (30). If optimists' leverage is unconstrained, their wealth share after transition and their leverage ratio are given by Eqs. (31) and (32). If their leverage ratio is constrained, these values are given by Eqs. (33) and (34). Optimists' wealth share evolves according to (35).

Our next result describes how macroprudential policy that tightens the leverage limit affects this equilibrium. This provides a useful benchmark for the next section where we assume macroprudential policy is imperfect and investigate whether PMP can provide similar financial stability benefits.

Proposition 1. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the benchmark equilibrium without PMP,* $Q_1(\cdot) = Q^*$*. Fix a level* $\alpha \in (0,1)$ *that is associated with some binding leverage limit,* $\overline{\omega}_1(\alpha) \leq \omega_1^o(\alpha, \infty)$*. Decreasing the leverage limit increases optimists' wealth share after a transition to recession:* $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$*. It also slows down the growth rate of optimists' wealth share if the boom persists,* $\frac{d(\dot{\alpha}_{t,1}/\alpha_{t,1})}{d\overline{\omega}_1(\alpha)} < 0$*.*

For a sketch proof (completed in Appendix A.3), note that optimists' wealth decline after transition is increasing in their leverage ratio, $\omega_1 - 1$ [cf. Eq. (34)]. Tightening the leverage limit reduces optimists' leverage ratio, $\tilde{\omega}_1 - 1 < \omega_1 - 1$, which in turn mitigates their wealth decline. This increases the price level in the recession, Q_2, which further boost optimists' wealth. In equilibrium, optimists' wealth share and the asset price in the recession settle at a higher level, $\alpha_2(\alpha, \tilde{\omega}_1) > \alpha_2(\alpha, \overline{\omega}_1)$ and $Q_2(\alpha_2(\alpha, \tilde{\omega}_1)) > Q_2(\alpha_2(\alpha, \overline{\omega}_1))$. The left panel of Figure 1 (in the introduction) illustrates the virtuous cycle that results from tightening the leverage limit.

Recall that increasing optimists' wealth share in the recession internalizes aggregate demand externalities [cf. Eq. (29)]. Therefore, Proposition 1 illustrates how macroprudential policy that tightens the leverage limit *can* improve welfare. At the same time, the welfare effects do not follow immediately because tightening the leverage limit also slows down the growth of optimists' wealth share if the recession is not realized, as illustrated by the last part of Proposition 1. In a dynamic setting, optimists' wealth share can also be useful in future recessions and thus macroprudential policy involves a trade-off. We investigate this trade-off in Caballero and Simsek (2017), where we show that the benefits from an immediate transition to recession often dominate the costs from worsening future recessions (in view of discounting). In particular, we show that (under regularity conditions and starting from a no-policy benchmark) adopting some macroprudential policy improves welfare.

5. Prudential monetary policy

We now assume that macroprudential policy is inflexible: the planner cannot change the existing leverage constraints. Instead, we introduce our main ingredient and allow monetary policy in the boom state to be used for prudential purposes. We start by establishing a negative result: when there is no leverage limit, PMP is useless because optimists endogenously change their risk taking to undo the prudential benefits. We then consider the case with a leverage limit and establish that, when there is *some* leverage limit, monetary policy can replicate the prudential effects of tightening this limit. Specifically, our main result establishes that, up to a first order, the welfare effects of PMP are the same as the effects of directly tightening the leverage limit.

Formally, suppose that in the boom state the planner does not follow the rule in (13) but instead sets the interest rate to target an asset price level, $Q_{t,1}$, which might be lower than the efficient level, $Q_{t,1} \leq Q^*$. We assume the planner's price target can be written as a function of optimists' wealth share:

$$Q_{t,1} = Q_1(\alpha_{t,1}) \leq Q^*.$$

We denote the equilibrium variables as functions of the PMP *function* (in addition to the earlier variables): $\alpha_2(\alpha, \overline{\omega}_1(\cdot), Q_1(\cdot))$ denotes optimists' wealth share after transition, when monetary policy is described by $Q_{t,1} = Q_1(\alpha_{t,1})$ for each t. We use the same notation as in the previous section to denote the equilibrium in the benchmark in which the planner follows the conventional output-stabilization policy: e.g., $\alpha_2(\alpha, \overline{\omega}_1)$ denotes the equilibrium when monetary policy is described by $Q_{t,1} = Q^*$ for each t.

5.1. No leverage limit

First consider the case without a leverage limit, $\overline{\omega}_1 = \infty$. In this case, we establish a *negative* result: PMP can only worsen the gap value (i.e., reduce welfare).

Proposition 2. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the case without a leverage limit, $\overline{\omega}_1 = \infty$, and some PMP, $Q_1(\cdot)$. Optimists' wealth share after transition and the evolution of their wealth share are the same as in the benchmark without prudential policy (in particular, Eq. (31) holds). The policy lowers the planner's gap value relative to the benchmark with conventional output-stabilization policy:*

$$w_1^{pl}(\alpha, \infty, Q_1) \leq w_1^{pl}(\alpha, \infty).$$

The first part of Proposition 2 says that PMP, by itself, does not affect the evolution of investors' wealth shares. The second part follows as a corollary. Since the policy does not affect wealth shares, it only affects the gap value through its impact on the asset price during the boom, $Q_{t,1}$ [cf. Eq. (28)]. Lowering $Q_{t,1}$ below Q^* makes factor utilization less efficient and decreases welfare: $W(Q_{t,1}) < W(Q^*)$ when $Q_{t,1} < Q^*$. Put differently, the policy has no benefits, but it has some costs due to low asset prices and inefficient factor utilization in the boom state.

The key step to our argument is that the policy does *not* affect optimists' wealth share after transition, $\alpha_2(\alpha, \infty, Q_1) = \alpha_2(\alpha, \infty) = \alpha \frac{\lambda_1^o}{\lambda_1(\alpha)}$ [cf. Eq. (31)]. To understand this feature, consider the equilibrium for an intermediate case, $\alpha \in (0, 1)$, and note that the policy affects optimists' equilibrium leverage ratio. In particular, we have the following version of Eq. (34):

$$\frac{\alpha_2(\alpha, \infty)}{\alpha} = 1 - (\omega_1^o(\alpha, \infty, Q_1) - 1)\left[\frac{Q_1(\alpha)}{Q_2(\alpha_2(\alpha, \infty))} - 1\right].$$

Note that a decline in $Q_1(\alpha)$ *does* result in a smaller price drop after transition (the term inside the brackets). Therefore, the policy leaves optimists' wealth share after transition (α_2) unchanged because it induces optimists to *increase* their leverage ratio, $\omega_1^o(\alpha,\infty,Q_1) > \omega_1^o(\alpha,\infty)$. Put differently, the prudential effects of the policy are neutralized by an increase in optimists' risk taking. Optimists increase their leverage because they perceive the transition to recession as less risky due to a smaller asset price drop after the transition.

5.2. With leverage limit

The previous discussion suggests that PMP *can* affect investors' equilibrium exposures if optimists are constrained by *some* leverage limit. Consider a situation in which there is a limit that binds for optimists so that $\omega_1^o(\alpha,\overline{\omega}_1,Q_1) = \overline{\omega}_1(\alpha)$. Then, we have the following version of Eq. (34):

$$\frac{\alpha_2(\alpha,\overline{\omega}_1,Q_1)}{\alpha} = 1 - (\overline{\omega}_1(\alpha) - 1)\left[\frac{Q_1(\alpha)}{Q_2(\alpha_2(\alpha,\overline{\omega}_1,Q_1))} - 1\right]. \tag{36}$$

In this case, since $\overline{\omega}_1(\alpha)$ is fixed, a decline in $Q_1(\alpha)$ translates into an increase in optimists' wealth share after transition. By reducing asset prices during the boom, the planner reduces the price drop after a transition to recession, which supports optimists' balance sheets. The following result formalizes this intuition and shows that monetary policy can replicate the prudential effects of tightening the leverage limit.

Proposition 3. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the benchmark equilibrium without PMP, $Q_1(\cdot) = Q^*$. Fix a level $\alpha \in (0,1)$ that is associated with some leverage limit, $\overline{\omega}_1(\alpha) < \infty$ (that might or might not bind). Consider an alternative leverage limit $\tilde{\omega}_1(\cdot)$ that agrees with $\overline{\omega}_1(\cdot)$ everywhere except for α and that satisfies $\tilde{\omega}_1(\alpha) < \min(\overline{\omega}_1(\alpha), \omega_1^o(\alpha,\infty))$, and a PMP $\tilde{Q}_1(\cdot)$ that agrees with $Q_1(\cdot)$ everywhere except for α. Then:*

(i) There exists $\tilde{Q}_1(\alpha) < Q^*$ such that the PMP (with the original leverage limit) generates the same effect on optimists' wealth share after transition as the alternative leverage limit (without PMP):

$$\alpha_2\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right) = \alpha_2(\alpha,\tilde{\omega}_1).$$

Targeting a lower effective limit requires targeting a lower asset price, $\frac{\partial \tilde{Q}_1(\alpha)}{\partial \tilde{\omega}_1(\alpha)} > 0$.

(ii) PMP requires setting a higher interest rate than the benchmark without policy:

$$r_1^f\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right) > r_1^f(\alpha,\overline{\omega}_1).$$

Targeting a lower effective limit requires setting a higher interest rate, $\frac{\partial r_1^f(\alpha,\overline{\omega}_1,\tilde{Q}_1)}{\partial \tilde{\omega}_1(\alpha)} < 0$.

The first part of Proposition 3 shows that monetary policy can replicate the prudential effects of tightening the leverage limit that we established in Proposition 1. For a sketch proof (completed in Appendix A.4), note that optimists' wealth decline after a transition depends on the product of their (above-average) leverage and the price decline, $(\omega_1 - 1)\left[\frac{Q_1}{Q_2} - 1\right]$ [cf. Eq.

(36)]. Recall that tightening the leverage limit mitigates optimists' wealth decline by reducing their leverage ratio, $\tilde{\omega}_1 - 1 < \omega_1 - 1$. For a given asset price Q_2, monetary policy can achieve the same wealth decline for optimists at the leverage limit, $\omega_1 = \overline{\omega}_1$, by reducing the asset price decline, $\frac{\tilde{Q}_1}{Q_2} - 1 < \frac{Q^*}{Q_2} - 1$. This policy increases the price level in the recession, Q_2, which generates a similar virtuous cycle as a policy that directly tightens the leverage limit. The right panel of Figure 1 illustrates how PMP generates effects that are very similar to tightening the leverage limit.

In fact, the monetary authority can choose $\tilde{Q}_1$ so that optimists' wealth share and the equilibrium price in the recession settle *exactly* at the same level as if the regulator had tightened the leverage limit, $\alpha_2\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) = \alpha_2\left(\alpha, \tilde{\omega}_1\right)$ and $Q_2\left(\alpha_2\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right)\right) = Q_2\left(\alpha_2\left(\alpha, \tilde{\omega}_1\right)\right)$. Specifically, after substituting these expressions into Eq. (36), we characterize $\tilde{Q}_1$ as the unique solution to

$$\left(\overline{\omega}_1\left(\alpha\right) - 1\right)\left[\frac{\tilde{Q}_1}{Q_2\left(\alpha_2\left(\alpha, \tilde{\omega}_1\right)\right)} - 1\right] = \left(\tilde{\omega}_1\left(\alpha\right) - 1\right)\left[\frac{Q^*}{Q_2\left(\alpha_2\left(\alpha, \tilde{\omega}_1\right)\right)} - 1\right]. \tag{37}$$

Hence, $\tilde{Q}_1$ is the asset price that replicates optimists' wealth decline after accounting for the endogenous price adjustment in the recession.

The second part of Proposition 3 shows that PMP requires raising the interest rate above the conventional policy benchmark with output stabilization. As expected, targeting a lower asset price requires a higher interest rate. This result offers an alternative interpretation for how PMP works. Recall that, if there is an instantaneous transition to the recession, then the interest rate will decline to zero with or without PMP, $r_2^f\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) = r_2^f\left(\alpha, \overline{\omega}_1\right) = 0$. Hence, by increasing the interest rate during the boom, PMP increases the size of the interest rate *cut* in case there is a transition to recession, $r_1^f - r_2^f$. For a given level of Q_2, this reduces the asset price *decline* after transition to recession, Q_1/Q_2. A smaller asset price decline supports optimists' wealth share after transition, α_2, and increases the asset price level Q_2 (which triggers the virtuous cycle described earlier). Thus, the policy can be thought of as increasing the interest rate to create room for an interest rate *cut* and mitigate the impact of negative asset price shocks in the future.

Proposition 3 is essentially static: it considers a policy change at a particular instant while leaving the policy at other times unchanged. This is useful for illustrating how PMP works, but it does not have an impact on the *dynamic* equilibrium. In addition, since PMP has costs as well as benefits, there is the remaining question of how it affects welfare. We next present our main result, which generalizes Proposition 3 to a dynamic setting and shows that the welfare effects of prudential policy are also (locally) equivalent to tightening the leverage limit.

To state the result, we parameterize the leverage limit function, $\overline{\omega}\left(\alpha, l\right)$ where $l \in L \subset \mathbb{R}_+$, and lower levels of l correspond to a tighter leverage limit, $\frac{\partial \overline{\omega}(\alpha, l)}{\partial l} > 0$ for $\alpha \in (0, 1)$. An example is the simple leverage limit function

$$\overline{\omega}_1(\alpha, l) = l \text{ with } l \in L = (1, \infty). \tag{38}$$

Here, the leverage limit doesn't depend on α and a lower l corresponds to a tighter limit for all α. Whenever we parameterize the leverage limit function, we simplify the notation by denoting the corresponding equilibrium variables with $\alpha_2(\alpha, l, Q_1)$ (as opposed to $\alpha_2(\alpha, \overline{\omega}_1(\cdot, l), Q_1)$).

Proposition 4. *Suppose Assumptions 1-2 and A1-A3 hold. Consider the case with some leverage limit function, $\overline{\omega}_1(\alpha, l)$, parameterized so that lower levels of l correspond to a tighter limit.*

(i) For each $\tilde{l} < l$ in a sufficiently small neighborhood of l, there exists a PMP, denoted by $Q_1\left(\cdot, \tilde{l}\right)$, such that optimists' equilibrium wealth share after transition is the same as when the leverage limit is given by $\overline{\omega}_1\left(\alpha, \tilde{l}\right)$ without PMP:

$$\alpha_2\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right) = \alpha_2\left(\alpha, \tilde{l}\right) \text{ for each } \alpha \in (0,1).$$

(ii) For small policy changes, the welfare effects of PMP are the same as the welfare effects of tightening the leverage limit directly:

$$\frac{dw_1^{pl}\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right)}{d\tilde{l}}|_{\tilde{l}=l} = \frac{dw_1^{pl}\left(\alpha, \tilde{l}\right)}{d\tilde{l}}|_{\tilde{l}=l}. \tag{39}$$

The first part of Proposition 4 follows from a similar analysis as in Proposition 3. In particular, for each $\alpha \in (0,1)$, the price level $Q_1\left(\alpha, \tilde{l}\right) = \tilde{Q}_1$ corresponds to the policy that replicates the prudential effects of the tighter leverage limit, $\overline{\omega}_1\left(\alpha, \tilde{l}\right) = \tilde{\omega}_1$, given the current limit $\overline{\omega}_1(\alpha, l)$.[6]

The second part characterizes the welfare effects of PMP for small amounts of effective tightening. For a sketch proof, note that the policies $\tilde{l}$ and $Q_1\left(\cdot, \tilde{l}\right)$ lead to identical equilibrium allocations except for the asset price in the boom state. Using this observation and the definition of the gap value in (28), the welfare difference between the two policies can be written as

$$w_1^{pl}\left(\alpha, l, Q_1\left(\cdot, \tilde{l}\right)\right) - w_1^{pl}\left(\alpha, \tilde{l}\right) = \int_0^\infty e^{-\left(\rho + \lambda_1^{pl}\right)t}\left(W\left(Q_1\left(\alpha_{t,1}, \tilde{l}\right)\right) - W\left(Q^*\right)\right) dt. \tag{40}$$

Here, $\alpha_{t,1}$ denotes optimists' wealth share when the economy starts with $\alpha_{0,1} = \alpha$, follows policy $\tilde{l}$, and reaches time t without transitioning into recession. Since $W(Q_{t,1}) < W(Q^*)$ for $Q_{t,1} < Q^*$, this expression implies that PMP always yields lower welfare than the equivalent tightening of the leverage limit. However, since $W(Q_{t,1})$ is maximized at $Q_{t,1} = Q^*$, these

[6] One difference from Proposition 3 is that the policy's effect on the interest rate is more complicated because the price drift $\dot{Q}_{t,1}$ is not necessarily zero. This non-zero drift affects the equilibrium return to capital [cf. Eq. (18)] and thus the equilibrium interest rate. As long as $\tilde{l}$ is in a neighborhood of l, this effect is small and the interest rate in the boom state remains strictly positive (in particular, the policy doesn't violate the zero lower bound). In fact, in the numerical simulations (described below), PMP increases the interest rate.

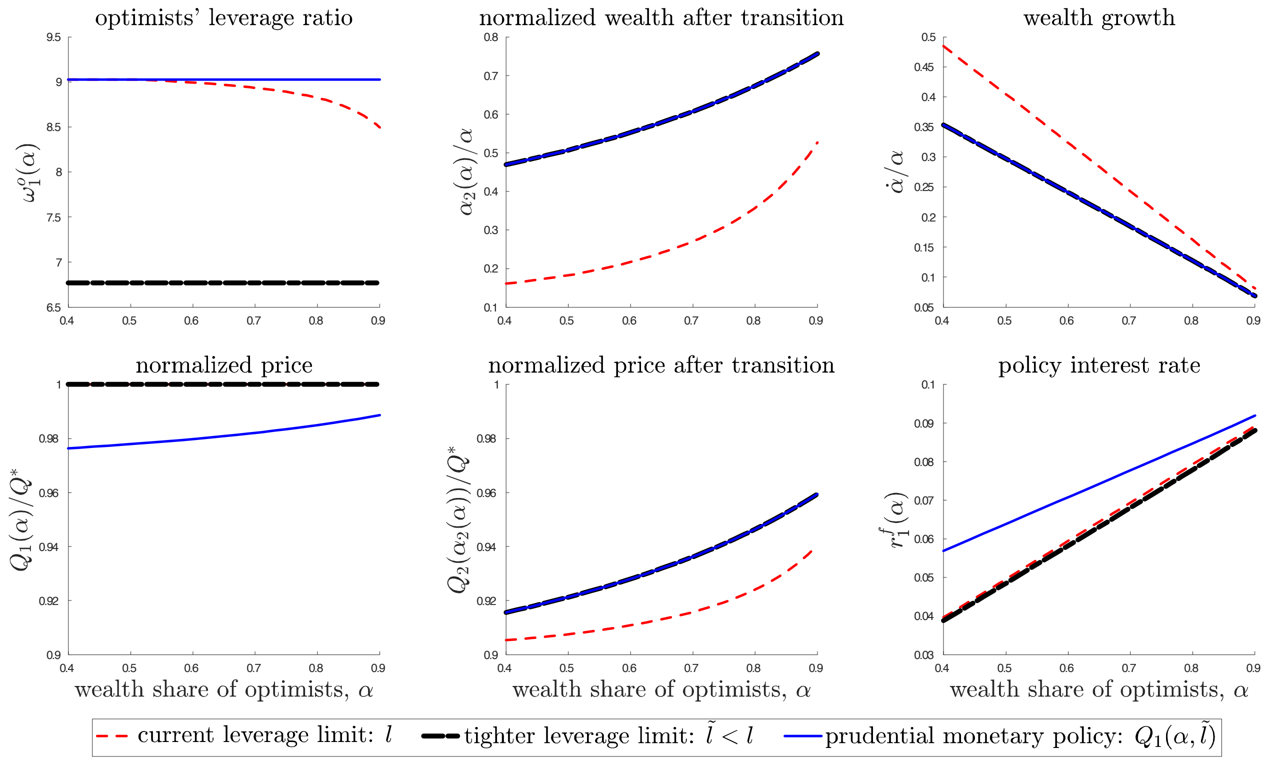

Figure 3: Equilibrium functions in the boom state $s = 1$ for different specifications of the leverage limit and PMP.

welfare differences are second order when the prudential policy is used in small doses (so that $Q_{t,1}$ remains close to Q^*). Therefore, as formalized by Eq. (39), the two policies have identical first-order effects on welfare.

5.3. Numerical illustration

We next illustrate the effects of PMP with a numerical example. Suppose optimists' and pessimists' beliefs about the probability of a transition to recession are given by $\lambda_1^o = 0.09 < \lambda_1^p = 0.9$ and the remaining parameters are as described in Appendix A.6. We work with the simple leverage limit function in (38). We assume the current (market-imposed) limit barely binds when optimists have half of the wealth share. This amounts to setting: $l = \omega_1^o(0.5, \infty) = 9.03$. The planner would like to tighten this constraint by a quarter, $\tilde{l} = 0.75l = 6.77$, but she cannot control the leverage limit directly. Instead, the planner implements the replicating prudential policy, $Q_1\left(\alpha, \tilde{l}\right)$.

Figure 3 plots the equilibrium functions for three different policy specifications over the range $\alpha \in [0.4, 0.9]$. The red dashed lines correspond to the case with the current leverage limit l but no prudential policy of any kind. The black dash-dotted lines correspond to tightening the leverage limit directly, $\tilde{l} = 0.75l$. Finally, the blue solid lines correspond to implementing this tightening via PMP, $Q_1\left(\alpha, \tilde{l}\right)$.

The top left panel illustrates optimists' leverage ratio as a function of their wealth share for each specification. Optimists have an above-average leverage ratio. The current (market-imposed) leverage limit restricts optimists' leverage ratio only slightly (not visible in the figure). The proposed tightening would restrict their leverage ratio considerably more. PMP *raises* optimists' leverage ratio (over the range $\alpha > 0.5$) as it pushes them against the leverage limit.

The top middle panel illustrates optimists' wealth share after transition normalized by their current wealth share, $\alpha_2(\alpha)/\alpha$. Optimists' wealth share declines after transition, $\alpha_2(\alpha)/\alpha < 1$. PMP replicates the effect of tightening the leverage limit and therefore increases optimists' wealth share after transition. The top right panel illustrates that this effective tightening slows down the growth of optimists' wealth share if there is no transition.

The bottom left panel illustrates the equilibrium asset price in the boom state normalized by the efficient level. The leverage limit (its current level or hypothetical tightening) leaves the asset price equal to its efficient level. In contrast, PMP reduces the asset price by around 2%. This relatively small decline is able to replicate the effects of a large reduction in optimists' leverage ratio because optimists' initial leverage ratio is high. With high and constrained leverage, small changes in asset prices have large effects on optimists' balance sheets [cf. (36)].

The bottom middle panel illustrates the price after a transition to recession normalized by the efficient level. PMP increases the asset price during the recession. We can gain intuition for this result by comparing this panel with the bottom left panel. By lowering the asset price during the boom, PMP reduces the *asset price decline* after a transition to recession. This smaller decline supports optimists' balance sheets and thus improves the *asset price level* during the recession by around 2%.

The bottom right panel illustrates the equilibrium interest rate. The leverage limit reduces the policy interest rate because it reduces optimists' effective asset demand. In contrast, PMP increases the policy interest rate (by less than 2 percentage points). This reduces the asset price, as illustrated by the bottom left panel, which results in a smaller asset price decline when there is a transition to recession. Equivalently, by raising the interest rate, monetary policy creates room to mitigate the asset price *decline* that results from negative shocks.

Figure 4 simulates the equilibrium variables over time (for each policy specification) for a particular initial wealth share for optimists, α_0, and a particular realization of uncertainty. We take $\alpha_0 = 0.85$, and we consider a path in which the economy transitions into the recession at $t = 0.2$ and recovers from the recession at $t = 0.6$ (other choices lead to qualitatively similar effects). The plots illustrate that PMP raises the asset price in the recession at the cost of reducing it in the boom. In this example, the increase in the asset price level during the recession is greater than the required decline during the boom, but this is not always the case. Regardless of the relative magnitudes, the policy improves welfare (as we will show) because the asset-price increase in the recession generates first-order benefits, whereas the asset-price decline in the boom generates second-order welfare losses.

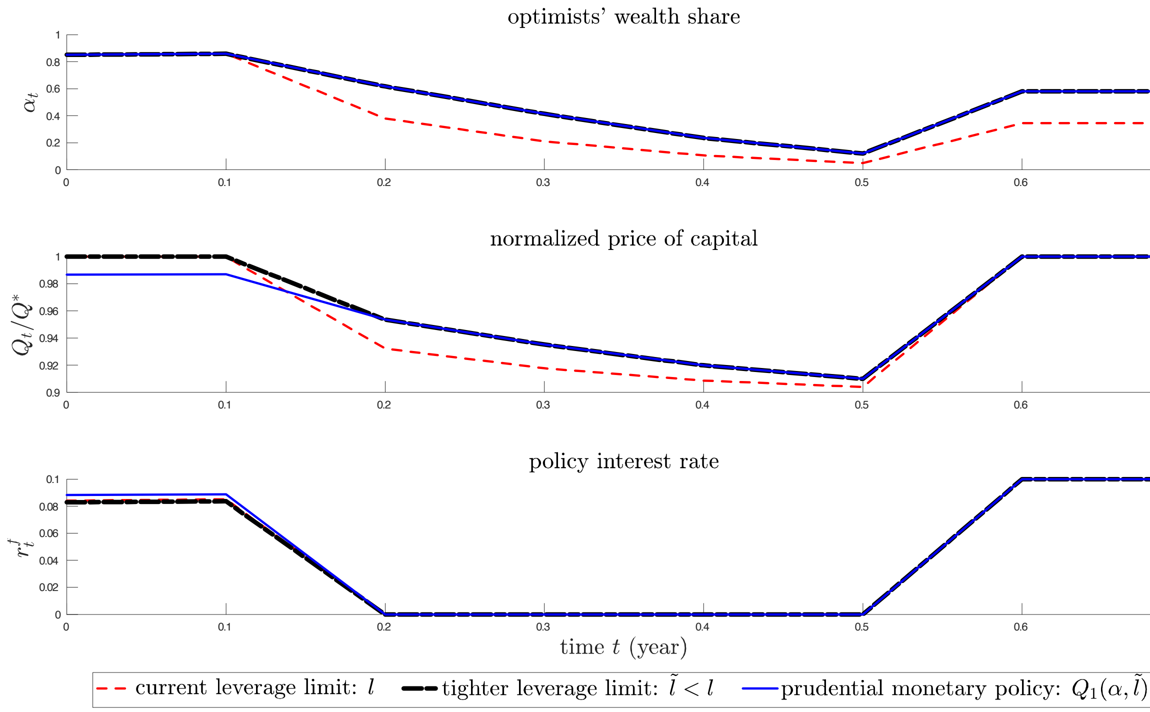

Figure 4: Simulation of the equilibrium path starting with $a_0 = 0.85$ and $s = 1$ for different specifications of the leverage limit and PMP.

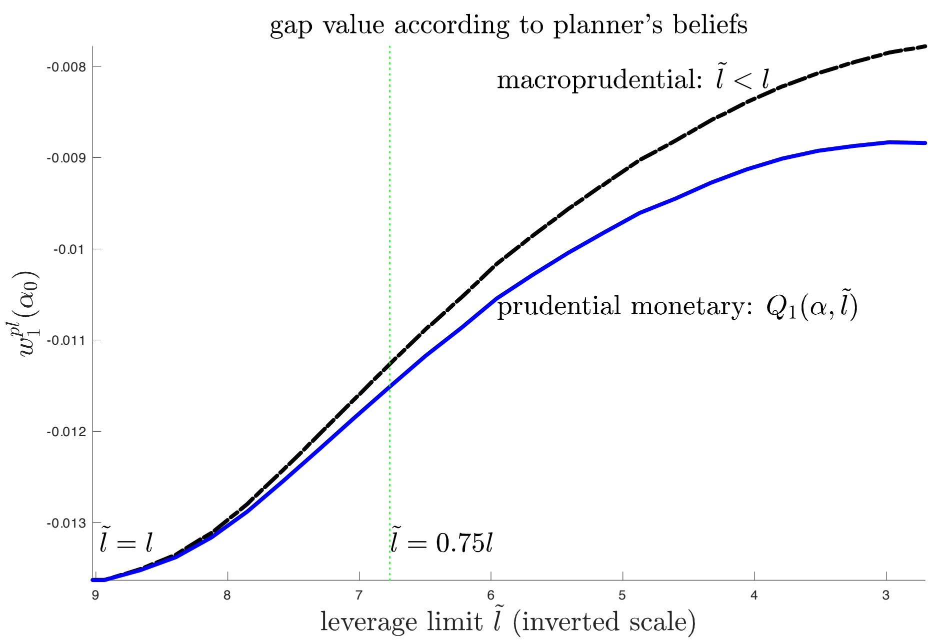

Figure 5: The planner's gap value as a function of the effective leverage ratio starting with $a_0 = 0.85$ and $s = 1$ for a direct tightening (dashed line) and an equivalent tightening via PMP (solid line).

Figure 5 illustrates the welfare effects of the policy by plotting the planner's gap value function, $w_1^{pl}(\alpha_0)$ [cf. Eq. (28)]. We take the planner's beliefs to be the average of optimists' and pessimists' beliefs, $\lambda_s^{pl} = (\lambda_s^o + \lambda_s^p)/2$. The black dash-dotted line in Figure 5 illustrates that, if feasible, a direct tightening of the leverage limit would improve the gap value. The solid blue line illustrates that an indirect tightening via PMP also increases the gap value. In fact, for small policy changes, PMP has the same welfare impact as a direct tightening, illustrating the second part of Proposition 4. This can be seen graphically in Figure 5 by comparing the gap values at the point corresponding to the leverage tightening studied above (which we highlight with the vertical dotted line). For small policy changes, welfare losses from the asset price decline during the boom are second order. As the (desired) limit is tightened further, these welfare losses grow larger and PMP becomes less desirable compared to a direct tightening.

6. Optimal prudential monetary policy

So far, we have established that monetary policy can have prudential benefits by effectively tightening an existing leverage limit. In this section, we analyze the determinants of optimal PMP in our setting. We first characterize the optimal prudential policy as the solution to a recursive optimization problem. We then solve the problem numerically and investigate the comparative statics of optimal policy.

For each α, suppose the planner sets an arbitrary price level $Q_1 \leq Q^*$ subject to the restriction that the price level weakly declines after the transition. Given Q_1, optimists' wealth share after transition is determined by the function $\alpha_2(\alpha, \overline{\omega}_1, Q_1) \in [0,1]$. This is a continuous and piecewise differentiable function that is equal to $\alpha_2(\alpha, \infty)$ if optimists' leverage limit does not bind (that is, if $\omega_1^o(\alpha, \infty, Q_1) < \overline{\omega}_1(\alpha)$) and is equal to the solution to (36) if the limit binds. Using this notation, we can recursively formulate the planner's optimization problem in the boom state $s = 1$ as:

$$\begin{aligned} \left(\rho + \lambda_1^{pl}\right) w_1^{pl}(\alpha) &= \max_{Q_1} W(Q_1) - W(Q^*) + \frac{dw_1^{pl}(\alpha)}{d\alpha}\dot{\alpha} + \lambda_1^{pl} w_2^{pl}(\alpha_2) \\ \text{where } \dot{\alpha} &= \frac{\alpha(1-\alpha)\lambda_1^p}{1-\alpha_2}\left(1 - \frac{\alpha_2}{\alpha}\right) \\ \alpha_2 &= \alpha_2(\alpha, \overline{\omega}_1, Q_1) \\ \text{and } Q_1 &\in \left[Q_2(\alpha_2(\alpha, \overline{\omega}_1, Q_1)), Q^*\right]. \end{aligned} \tag{41}$$

Here, the second line uses Eq. (35) to describe the evolution of optimists' wealth share absent a transition, $\dot{\alpha} = \frac{d\alpha_{t,1}}{dt}$, as a function of their induced wealth share after transition, $\alpha_2 = \alpha_{t,2}$ (as well as their current wealth share, $\alpha = \alpha_{t,1}$).[7]

The analytical solution to problem (41) is complicated in part because there might be a

[7] In problem (41), we ignore the zero lower bound constraint on the interest rate. In numerical solutions (described subsequently), we check and verify that this constraint doesn't bind at the optimal solution.

discontinuity in the optimal policy function.[8] However, it is straightforward to solve problem (41) numerically. Moreover, we can glean some intuition by considering the local optimality conditions. Specifically, for an interior solution $Q_1 \in (Q_2, Q^*)$, the optimality condition (for decreasing Q_1 further) can be written as:

$$\begin{aligned} \frac{dW\left(Q_1\right)}{dQ_1} &= \frac{d\alpha_2}{d\left(-Q_1\right)}\left[\lambda_1^{pl}\frac{dw_2^{pl}\left(\alpha_2\right)}{d\alpha_2}+\frac{d\dot{\alpha}}{d\alpha_2}\frac{dw_1^{pl}\left(\alpha\right)}{d\alpha}\right] \\ \text{where } \frac{d\dot{\alpha}}{d\alpha_2} &= -\lambda_1^{p}\frac{\left(1-\alpha\right)^2}{\left(1-\alpha_2\right)^2}. \end{aligned} \tag{42}$$

The left-hand side of Eq. (42) captures the costs of the policy via its impact on the output gap in period 1. This term is positive since $W'\left(Q_1\right) > 0$: decreasing the asset price in the boom exacerbates the output gap. The right-hand side captures the welfare effects of the policy via its impact on optimists' wealth share. We have $\frac{d\alpha_2}{d(-Q_1)} > 0$: lowering the asset price increases optimists' wealth share after transition. We also have $\frac{dw_2^{pl}(\alpha_2)}{d\alpha_2} > 0$: increasing optimists' wealth share after transition internalizes aggregate demand externalities and mitigates output gaps. Hence, the first term inside the brackets is positive and captures the static benefits of PMP. On the other hand, we also have $\frac{d\dot{\alpha}}{d\alpha_2} < 0$: if there is no transition, the policy slows down the accumulation of optimists' wealth share. Moreover, we have $\frac{dw_1^{pl}(\alpha)}{d\alpha} > 0$: the reduction in optimists' wealth share in the boom state widens output gaps in a future recession. Therefore, the second term inside the brackets is negative and captures the dynamic costs of PMP.

6.1. Numerical illustration

Figure 6 illustrates the optimal monetary policy corresponding to the numerical example in Section 5.2. As a benchmark, the red dashed lines illustrate the equilibrium without PMP but with the simple leverage limit $\overline{\omega}_1\left(\alpha, l\right) = l = 9.03$. Recall that this leverage limit is chosen so that (absent PMP) it binds for optimists when $\alpha < 0.5$ but not when $\alpha \geq 0.5$. The green dotted line in the left panel illustrates the minimum price decline necessary to make the leverage limit bind for optimists—price reductions smaller than this level have no prudential benefits as they are undone by endogenous risk adjustments by optimists.

The blue solid line in the left panel of Figure 6 illustrates the optimal price that solves problem (41). With this parameterization, the planner does not use monetary policy for prudential purposes when $\alpha < 0.33$. In this range, the leverage limit is already tight, and tightening it further via PMP does not create large enough benefits to compensate for the costs imposed by slowing down the accumulation of optimists' wealth share [cf. Eq. (42)]. In contrast, the planner

[8] This discontinuity emerges from the fact that, if the leverage limit doesn't bind absent policy ($\omega_1^o\left(\alpha, \infty, Q_1^*\right) < \overline{\omega}_1\left(\alpha\right)$), then prudential monetary policy requires a discontinuous decline in asset prices and output. In particular, there might be a threshold level of optimists' wealth share, $\overline{\alpha}$, where the planner is indifferent between setting $Q_1\left(\overline{\alpha}\right) < Q_1$ (and using the policy) and setting $Q_1\left(\overline{\alpha}\right) = Q_1^*$ (and not using the policy).

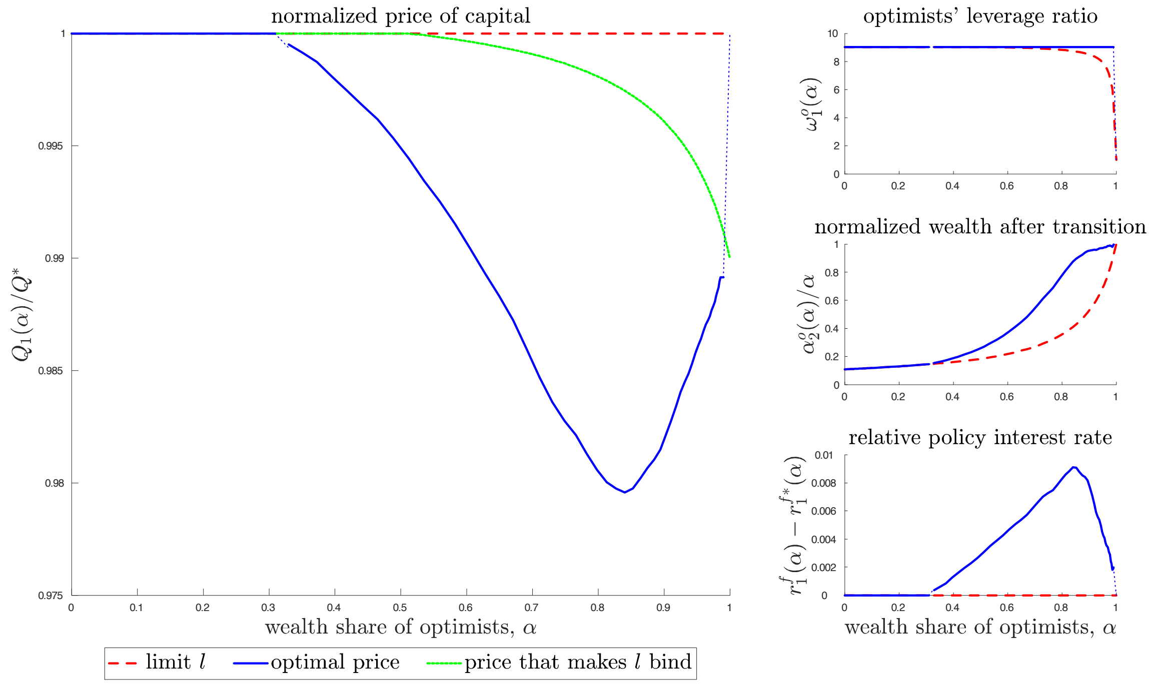

Figure 6: Equilibrium with optimal PMP (blue solid line) and without PMP (red dashed line) given the leverage limit l. The green dotted line in the left panel illustrates the minimum price decline necessary to make optimists' leverage limit bind.

uses PMP over the range $\alpha \in [0.33, 0.99]$. Moreover, the degree of tightening relative to the conventional policy benchmark is non-monotonic in the optimists' wealth share. In particular, the planner tightens the policy more as optimists' wealth share increases toward $\alpha = 0.85$ and tightens it less beyond this level. Hence the policy is most useful when optimists' wealth share lies in an intermediate range. Two forces make the policy relatively less attractive for large α. First, since optimal private leverage drops as α rises, the policy becomes costlier as the planner needs to reduce the price even further to make optimists' leverage limit bind and gain some traction (as illustrated by the green dotted line). Second, the policy is less useful because there is less speculation. In fact, for $\alpha \simeq 0.99$, these countervailing forces are strong enough that the planner stops using the policy altogether (as illustrated by the jump in the blue solid line).

Figure 7 illustrates the comparative statics of the optimal policy. To facilitate exposition, we describe the effects for a particular level of optimists' wealth share, $\alpha = 0.85$ (the same wealth share we considered in the previous section). The top panels display the change in the optimal price level as we vary a single parameter. The bottom panels display the change in the optimal interest rate relative to the conventional policy benchmark with output stabilization.

The left panels show the effect of changing the leverage limit, l. When the leverage limit is very loose, the planner does not use prudential policy because it is easily undone by optimists, illustrating Proposition 2. There is a threshold leverage limit below which the planner uses

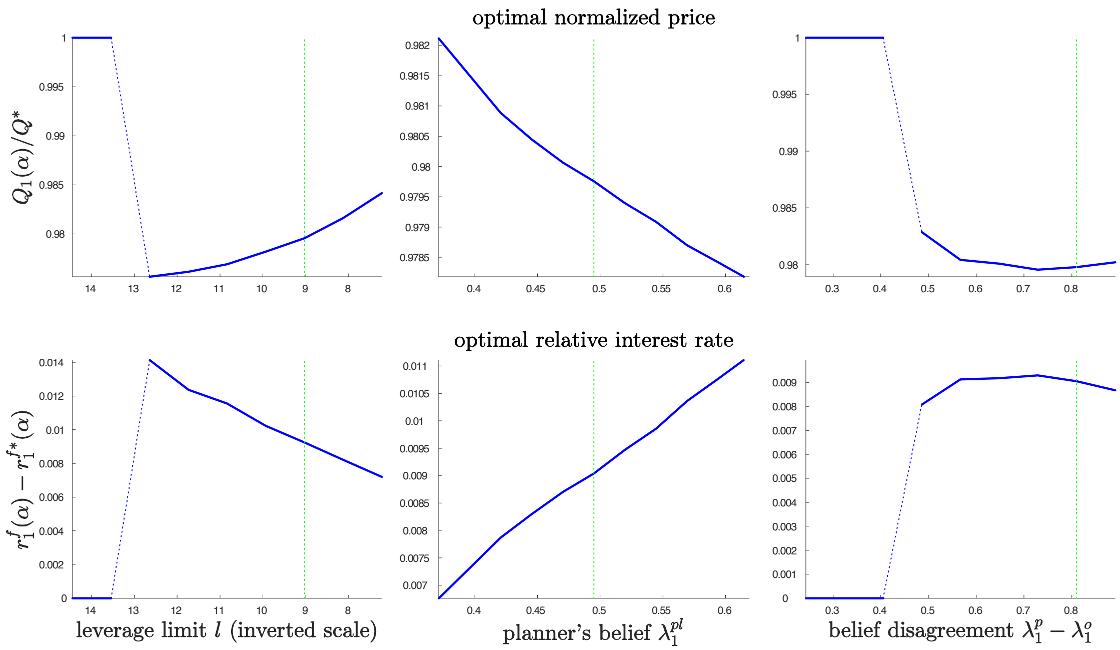

Figure 7: Comparative statics of the optimal PMP price level (top panels) and the interest rate (bottom panels) for $\alpha = 0.85$ with respect to changing the parameter on the x-axis. The vertical dotted lines illustrate the benchmark parameters (used in earlier figures).

monetary policy. Once the leverage limit is below this threshold, tightening it further makes the planner use PMP less. Hence, the leverage limit and PMP are *complements* in the high-l range but they become *substitutes* in the low-l range.

The middle panels illustrate the effect of changing the planner's belief about the probability of transition into recession, λ_1^{pl}. As expected, when the planner believes the recession is more likely, she utilizes PMP more and reduces the asset price by a greater amount.

The right panels show the effect of changing belief disagreements, $\lambda_1^p - \lambda_1^o$ (keeping the mean belief constant at $\frac{\lambda_1^p + \lambda_1^o}{2}$). With greater belief disagreements, the planner is more likely to utilize PMP. Intuitively, disagreements increase speculation (and optimists' risk-exposure), which makes PMP more useful. Conditional on using the policy, the planner does not change the intensity of the policy very much.[9] This insensitivity arises because, once the policy is used, it sets optimists against the leverage limit, which largely decouples equilibrium outcomes from the magnitude of belief disagreements.

[9] In particular, the main effect of greater disagreements is to reduce the threshold level of optimists' wealth share above which the planner uses prudential monetary policy (see Figure 6).

7. Prudential policies with "shadow banks"

In practice, a major concern with macroprudential policy is that there are lightly regulated institutions—typically referred to as shadow banks—that can circumvent leverage limits or other regulatory constraints. Stein (2013) noted that in these environments PMP might have an advantage over macroprudential policy "because it gets in all of the cracks." We next evaluate the performance of macroprudential policy and PMP in our model when some of the high-valuation agents face a looser leverage limit. We conclude that both policies remain useful but are weakened by the same general equilibrium forces that incentivize shadow banks to increase their leverage.

Specifically, suppose a subset of optimists are not subject to the leverage constraint, $\omega_{1,t} \leq \overline{\omega}_{1,t}$. We refer to these agents as *unregulated optimists*, and refer to the remaining fraction of optimists as *regulated optimists*. Recall that we view (regulated) optimists as the model counterpart to "banks" (see Remark 2). Therefore, unregulated optimists are the model counterpart to "shadow banks." The assumption that unregulated optimists face no leverage limit simplifies the analysis but is clearly extreme. Even if shadow banks can avoid all regulation, they may still be subject to market-based leverage constraints. Our assumption is only intended to qualitatively capture that shadow banks face looser leverage limits compared to banks.

We let $\beta \in (0,1)$ denote the relative fraction of optimists' wealth that is held by unregulated optimists. Hence, the wealth share of unregulated and regulated optimists is given by, respectively, $\alpha\beta$ and $\alpha(1-\beta)$. As before, the total wealth share of optimists (including both types) and pessimists is given by, respectively, α and $1-\alpha$. The rest of the model is unchanged.

To characterize the equilibrium, consider first the recession state $s=2$. Conditional on the total mass of optimists, α_2, the equilibrium is the same as before. This is because we assume optimists face no constraints from state 2 onwards, which implies there is no remaining functional difference between regulated and unregulated optimists. In particular, the equilibrium price in the recession can be written as $Q_{t,2} = Q_2(\alpha_{t,2})$, where $Q_2(\cdot)$ is the price function characterized earlier [cf. Eq. (26)].

Next consider the equilibrium in the boom state $s=1$. In this case, there are two state variables: the total mass of optimists, $\alpha \in (0,1)$, and the fraction of unregulated optimists, $\beta \in (0,1)$. Therefore, we denote the equilibrium variables as functions of two state variables, in addition to the leverage constrained policy and the PMP. In particular, $\alpha_2(\alpha, \beta, \overline{\omega}_1(\cdot), Q_1(\cdot))$ and $\beta_2(\alpha, \beta, \overline{\omega}_1(\cdot), Q_1(\cdot))$ denote, respectively, the total mass of optimists and the fraction of unregulated optimists that obtains if there is an instantaneous transition to recession. To simplify the notation, we suppress the dependence of these functions on some or all of their arguments as long as the appropriate arguments are clear from the context.

Much of our earlier analysis applies also in this setting. In particular, Eq. (22), which characterizes the growth rate of agents' wealth shares absent a state transition, applies for all agents. In the appendix, we solve the corresponding equations for regulated and unregulated optimists to obtain the dynamics of α and β as follows:

$$\begin{aligned}\frac{\dot{\alpha}}{\alpha} &= \lambda_1^p \frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2}{\alpha}\right), \\ \frac{\dot{\beta}}{\beta} &= \lambda_1^p \frac{1-\alpha}{1-\alpha_2}\frac{\alpha_2}{\alpha}\left(1-\frac{\beta_2}{\beta}\right).\end{aligned} \tag{43}$$

Given α_2, optimists' total wealth share follows the same equation as before (cf. Eq. (35)). Given β_2 and α_2, the relative wealth share of unregulated optimists follows a similar equation. Below, we will verify that the equilibrium features $\alpha_2 < \alpha$ and $\beta_2 < \beta$. Combining this observation with (43) implies $\dot{\alpha} > 0$ and $\dot{\beta} > 0$. Optimists' total wealth share (resp. unregulated optimists' relative wealth share) grow absent transition to recession, because these agents take on greater risk and earn a higher risk premium compared to pessimists (resp. regulated optimists).

It remains to characterize the functions, α_2, β_2. To this end, note that the portfolio optimality condition (21) holds as equality for unregulated optimists and as a weak inequality for regulated optimists. Combining these observations, we obtain:

$$\lambda_1^o \frac{\alpha(1-\beta)}{\alpha_2(1-\beta_2)} \geq \lambda_1^o \frac{\alpha\beta}{\alpha_2\beta_2} = \lambda_1^p \frac{1-\alpha}{1-\alpha_2}. \tag{44}$$

Note also that Eq. (20) , which relates agents' wealths share after transition to their leverage ratio, applies for all agents. Applying this condition for regulated and unregulated optimists, we obtain:

$$\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)} = 1-(1-\omega_1^{o,reg})\left(\frac{Q_1}{Q_2(\alpha_2)}-1\right) \tag{45}$$

$$\frac{\alpha_2\beta_2}{\alpha\beta} = 1-(1-\omega_1^{o,unreg})\left(\frac{Q_1}{Q_2(\alpha_2)}-1\right). \tag{46}$$

Given the current price level Q_1 and the price function after transition $Q_2(\alpha_2)$, the equilibrium functions for α_2, β_2 (as well as for $\omega_1^{o,reg}, \omega_1^{o,unreg}$) can be characterized by solving Eqs. (44 – 46).

Consider the case in which regulated optimists' leverage constraint binds (the other case is the same as in previous sections). In this case, we have $\omega_1^{o,reg} = \overline{\omega}_1$. Substituting this into Eq. (45), we obtain:

$$\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)} = 1-(\overline{\omega}_1-1)\left[\frac{Q_1}{Q_2(\alpha_2)}-1\right]. \tag{47}$$

As before, this expression describes regulated optimists' relative wealth share as a function of the leverage limit and the price drop after transition. Solving for β_2 from Eq. (44), and substituting into Eq. (47), we further obtain:

$$\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right) = 1-(\overline{\omega}_1-1)\left[\frac{Q_1}{Q_2(\alpha_2)}-1\right]. \tag{48}$$

This equation generalizes Eq. (34) (which we analyzed extensively in previous sections) to cases with $\beta > 0$. In particular, the equation characterizes α_2 given $Q_1, Q_2(\alpha_2)$ and $\overline{\omega}_1$.

Note also that the left-hand side of Eq. (48) is an increasing function of α_2. Hence, as before, the equation can be visualized as the intersection of two increasing relations between α_2 and Q_2. Under appropriate regularity conditions (relegated to the appendix), there is a unique intersection. The following result considers the benchmark case without PMP, $Q_1 = Q^*$, and establishes the comparative statics of the equilibrium with respect to the fraction of unregulated optimists, β. The result also establishes the comparative statics with respect to the leverage limit $\overline{\omega}_1$ and generalizes our earlier result about macroprudential policy (Proposition 1) to this setting.

Proposition 5. *Suppose Assumptions 1-2 and A1-A3 hold and that a fraction, $\beta \in (0,1)$, of optimists' wealth is held by unregulated optimists that face no leverage limits. Consider the benchmark equilibrium without PMP, $Q_1(\alpha) = Q^*$. Fix levels $\alpha, \beta \in (0,1)$ that are associated with some binding leverage limit, $\overline{\omega}_1(\alpha,\beta) < \omega_1^{o,reg}(\alpha,\beta,\infty)$. Absent transition to recession, α and β follow the dynamics in (43). After transition, α_2 is characterized as the solution to Eq. (48) and β_2 is characterized as the solution to (44). In equilibrium, $\alpha_2 < \alpha, \beta_2 < \beta$ and $\dot{\alpha} > 0, \dot{\beta} > 0$: optimists' total wealth share and unregulated optimists' relative wealth share shrink after transition to recession and grow absent transition. Moreover, α_2 satisfies the following comparative statics:*

(i) Increasing the relative wealth share of unregulated optimists, β, decreases optimists' wealth share after transition, $\frac{d\alpha_2(\alpha,\beta,\overline{\omega}_1(\cdot))}{d\beta} < 0$. In the limit as $\beta \to 1$, optimists' wealth share approaches its level in the equilibrium without leverage limits, $\alpha_2(\alpha,\infty)$.

(ii) Macroprudential policy that decreases the leverage limit increases optimists' wealth share after a transition to recession, $\frac{d\alpha_2(\alpha,\beta,\overline{\omega}_1(\cdot))}{d\overline{\omega}_1(\alpha,\beta)} < 0$. Increasing the relative wealth share of unregulated optimists, β, reduces the effectiveness of macroprudential policy, $\frac{\partial}{\partial\beta}\frac{d\alpha_2(\alpha,\beta,\overline{\omega}_1(\cdot))}{d\overline{\omega}_1(\alpha,\beta)} > 0$.

This result verifies the conventional wisdom that the presence of less regulated agents reduces the strength of macroprudential policy. The first part shows that, as the relative wealth share of unregulated optimists grows, optimists take on greater risk and their wealth share declines by a greater magnitude after transition to recession. The second part shows that (as long as some optimists are regulated, $\beta < 1$) macroprudential policy that tightens leverage limits mitigates the decline in optimists' wealth share but less so than in the earlier setting without unregulated optimists.

Next consider PMP that lowers the current asset price level, $Q_1(\alpha,\beta) \leq Q^*$. As illustrated by Eq. (47), PMP reduces regulated optimists' exposure to transition to recession. As illustrated by Eq. (48), this in turn increases the total mass of optimists after transition to recession, α_2. Eq. (48) further suggests that, as before, PMP affects the equilibrium in much the same way as a decline $\overline{\omega}_1$. The following result verifies this intuition and generalizes our main result showing that monetary policy can replicate the prudential effects of tightening a leverage limit (cf. Proposition 3).

Proposition 6. *Suppose Assumptions 1-2 and A1-A3 hold and that a fraction, $\beta \in (0,1)$, of optimists' wealth is held by unregulated optimists that face no leverage limits. Fix some $\alpha, \beta \in (0,1)$ and consider the setup of Proposition 3. In particular, consider an alternative leverage limit $\tilde{\omega}_1(\cdot)$ that agrees with $\overline{\omega}_1(\cdot)$ everywhere except for (α,β) and that satisfies $\tilde{\omega}_1(\alpha,\beta) < \min\left(\overline{\omega}_1(\alpha,\beta), \omega_1^{o,reg}(\alpha,\beta\infty)\right)$. Then:*

(i) There exists $\tilde{Q}_1(\alpha,\beta) < Q^*$ such that the PMP (with the original leverage limit) generates the same effect on regulated and unregulated optimists' wealth shares after transition as the alternative leverage limit (without PMP):

$$\alpha_2\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right) = \alpha_2\left(\alpha,\beta,\tilde{\omega}_1\right) \text{ and } \beta_2\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right) = \beta_2\left(\alpha,\beta,\tilde{\omega}_1\right).$$

Targeting a lower effective limit requires targeting a lower asset price, $\frac{\partial\tilde{Q}_1(\alpha,\beta)}{\partial\tilde{\omega}_1(\alpha,\beta)} > 0$.

(ii) PMP requires setting a higher interest rate than the benchmark without policy:

$$r_1^f\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right) > r_1^f\left(\alpha,\beta,\overline{\omega}_1\right).$$

Targeting a lower effective limit requires setting a higher interest rate, $\frac{\partial r_1^f\left(\alpha,\beta,\overline{\omega}_1,\tilde{Q}_1\right)}{\partial\tilde{\omega}_1(\alpha,\beta)} < 0$.

The sketch-proof of this result is the same as in Proposition 3. In particular, the monetary authority can choose $\tilde{Q}_1$ so that optimists' total wealth share and the equilibrium price in the recession settle at the same level as if the regulator had directly tightened the leverage limit. In fact, conditional on optimists' wealth share α_2, the replicating $\tilde{Q}_1$ that the planner needs to set is characterized as the solution to the same equation (37) as in our earlier analysis.

7.1. Numerical illustration

We next illustrate numerically the effects of macroprudential policy and PMP in the presence of unregulated optimists. Consider the same example we analyzed in Section 5.3. In particular, the current leverage limit barely binds when optimists have half of the wealth share. The planner would like to tighten the existing limit by a quarter, $\tilde{l} = 0.75l$. However, she cannot control the leverage limit directly. Instead, the planner implements the replicating prudential policy, $Q_1\left(\alpha,\beta,\tilde{l}\right)$.

Figure 8 plots the equilibrium functions for three different policy specifications over the range $\alpha \in [0.4, 0.9]$ and $\beta \in [0,1]$. The lines corresponding to $\beta = 0$ match the earlier equilibria without unregulated optimists (also plotted in Figure 3). The rest of the surfaces illustrate the effect of unregulated optimists.

First consider the effect of macroprudential policy that tightens leverage limits: specifically, compare the benchmark with the current limit (illustrated with red lines) with a direct tightening

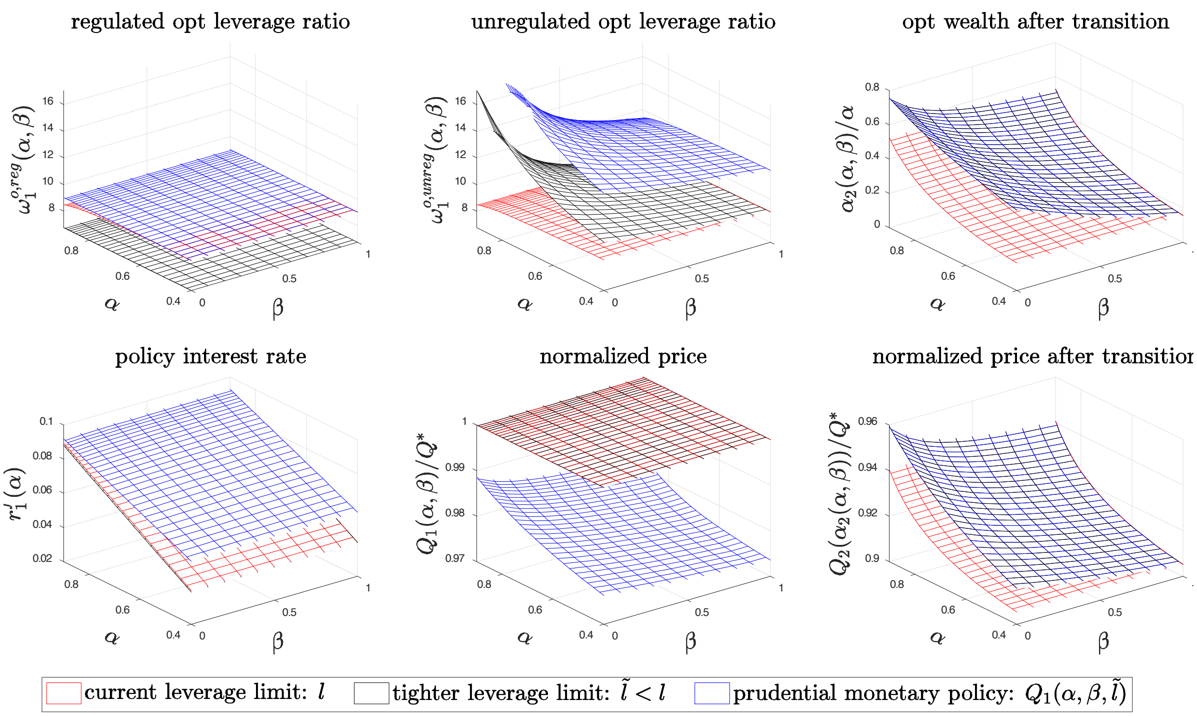

Figure 8: Equilibrium functions in the boom state $s = 1$ with unregulated optimists for different specifications of the leverage limit and PMP. β is the fraction of optimists' wealth held by unregulated optimists.

of the limit (illustrated with black lines). The top two left panels show regulated and unregulated optimists' leverage ratios, respectively. In the benchmark, regulated and unregulated optimists have similar leverage ratios (since the leverage limit barely binds). The proposed tightening of the leverage limit reduces regulated optimists' leverage ratio while *raising* unregulated optimists' leverage ratio. Intuitively, tightening the leverage limit reduces financial stability risk, since it increases asset prices after transition to recession. Unregulated optimists respond by taking greater risks.

The top right panel illustrates optimists' wealth share after transition to recession. Macroprudential policy improves optimists' wealth share in the recession but less so than in the case without unregulated optimists ($\beta = 0$), illustrating Proposition 5. Intuitively, since unregulated optimists respond to the policy by increasing their risks, they reduce (but do not fully eliminate) the effectiveness of macroprudential policy. Consequently, macroprudential policy improves asset prices in the recession but less so than in the case without unregulated optimists.

Next consider the PMP (illustrated with blue lines) that replicates the prudential effects of a direct tightening of the leverage limit. The two panels in the bottom left show that PMP achieves this outcome by increasing the interest rate and lowering asset prices during the boom, illustrating Proposition 6. The two panels in the top left show that PMP increases the leverage ratio of regulated optimists (as it pushes them against the leverage limit) *and* the leverage ratio of unregulated optimists. In fact, unregulated optimists respond by increasing their leverage ratio even more than when the planner directly tightens the leverage limit. These agents obtain the same wealth share after transition, $\alpha_2\beta_2$, as in direct tightening (see Proposition 6). However, they now achieve this outcome by taking on greater leverage since the price drop after transition is smaller (see Eq. (46)).

These results illustrate that, when some high-valuation agents are lightly regulated, PMP can still replicate the financial stability benefits of macroprudential policy. However, in our setting, PMP is subject to similar limitations as macroprudential policy: less regulated agents respond to the policy by increasing their leverage and risk taking. This finding is consistent with recent empirical evidence showing that a contractionary monetary policy shock *increases* lending by shadow banks (see Elliott et al. (2019); Drechsler et al. (2019)).

8. Final Remarks

We propose a model of asset price booms with speculation that may justify using PMP to reduce the severity of future recessions. PMP aims to reduce the social cost of concentrating risk in leveraged, high-valuation agents ("optimists" or "banks"). The policy achieves this goal by lowering the asset price level during the boom, which reduces the asset price decline after a transition to recession. This reduction supports highly-levered agents' balance sheets in the recession, which in turn raises asset prices (and hence further reduces the price drop) and softens the recession.

An equivalent interpretation is that PMP raises the interest rate to increase the available "ammunition" for the next recession. This concept has little meaning in most macro models where all that matters during a recession is the level of interest rates. By contrast, our framework emphasizes the importance of the size of interest rate cuts as the economy transitions from boom to recession. A larger interest rate cut is useful because it mitigates the asset price *decline* as the economy transitions to recession. A smaller asset price decline is preferable because it improves highly-levered agents' wealth share, which is a key state variable that determines the severity of the recession.

Our main insight can be applied beyond the specific binary-state context of our model. For example, in practice, large recessions are often preceded by minor slowdowns, at which time central banks need to decide how quickly to cut interest rates. Our analysis suggests that, if the slowdown is associated with significant financial speculation, then it may be worth delaying interest rate cuts. By doing so, the central bank effectively keeps its ammunition for a larger recession in which monetary policy becomes constrained.

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A. Appendix: Omitted derivations

This appendix presents the derivations and proofs omitted from the main text.

A.1. Omitted derivations in Section 2

A.1.1. Recursive formulation of the portfolio problem

We start by deriving the investors' optimality conditions. Recall that the investor's portfolio problem is given by (7). The HJB equation corresponding to this problem is

$$\begin{aligned}\rho V_{t,s}^{i}\left(a_{t,s}^{i}\right) = \max_{c,\omega} \log c + \frac{\partial V_{t,s}^{i}}{\partial a}\left(a_{t,s}^{i}\left(r_{t,s}^{f} + \omega\left(r_{t,s} - r_{t,s}^{f}\right)\right) - c\right) \\ + \frac{\partial V_{t,s}^{i}\left(a_{t,s}^{i}\right)}{\partial t} + \lambda_{s}^{i}\left(V_{t,s'}^{i}\left(a_{t,s}^{i}\left(1 + \omega\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right)\right) - V_{t,s}^{i}\left(a_{t,s}^{i}\right)\right) \\ \text{s.t. } \omega \leq \overline{\omega}_{t,1} \text{ if } s = 1.\end{aligned} \tag{A.1}$$

In view of log utility, the solution has the functional form

$$V_{t,s}^{i}\left(a_{t,s}^{i}\right) = \frac{\log\left(a_{t,s}^{i}/Q_{t,s}\right)}{\rho} + v_{t,s}^{i}. \tag{A.2}$$

The first term in the value function captures the effect of holding a greater capital stock (or greater wealth), which scales the investor's consumption proportionally at all times and in all states. The second term, $v_{t,s}^{i}$, is the normalized value function when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^{i} = Q_{t,s}$). This functional form also implies

$$\frac{\partial V_{t,s}^{i}}{\partial a} = \frac{1}{\rho a_{t,s}^{i}}.$$

The first order condition for c then implies Eq. (14) in the main text. The first order condition for ω implies

$$\frac{\partial V_{t,s}^{i}}{\partial a} a_{t,s}^{i}\left(r_{t,s} - r_{t,s}^{f}\right) + \lambda_{s}^{i}\frac{\partial V_{t,s'}^{i}\left(a_{t,s'}^{i}\right)}{\partial a} a_{t,s}^{i}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \geq 0,$$

with inequality only if $s = 1$ and $\omega = \overline{\omega}_{t,1}$. After substituting for $\frac{\partial V_{t,s}^{i}}{\partial a}$ and $\frac{\partial V_{t,s'}^{i}}{\partial a}$ and rearranging terms, this relation implies

$$r_{t,s} - r_{t,s}^{f} + \lambda_{s}^{i}\frac{a_{t,s}^{i}}{a_{t,s'}^{i}}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} \geq 0,$$

with inequality only if $s = 1$ and $\omega = \overline{\omega}_{t,1}$. After substituting $a_{t,s}^{i} = \alpha_{t,s}^{i} Q_{t,s} k_{t,s}$ [cf. Eq. (19)], this gives Eq. (21) in the main text.

A.1.2. Evolution of investors' wealth share

We next derive the evolution of investors' wealth shares. After substituting optimal consumption from (14) into the budget constraint in problem (7), type i investors' wealth evolves according to

$$\frac{da^i_{t,s}/dt}{a^i_{t,s}} = r^f_{t,s} + \omega^i_{t,s}\left(r_{t,s} - r^f_{t,s}\right) - \rho.$$

Combining this with Eq. (9), aggregate wealth evolves according to

$$\frac{d\left(Q_{t,s}k_{t,s}\right)/dt}{Q_{t,s}k_{t,s}} = r^f_{t,s} + \left(r_{t,s} - r^f_{t,s}\right) - \rho.$$

Combining these expressions with $\alpha^i_{t,s} = \frac{a^i_{t,s}}{Q_{t,s}k_{t,s}}$ [cf. Eq. (19)], we obtain:

$$\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} = \left(\omega^i_{t,s} - 1\right)\left(r_{t,s} - r^f_{t,s}\right). \tag{A.3}$$

Next recall that the portfolio optimality condition (21) holds with equality for pessimists. Applying this equation, we obtain:

$$r_{t,s} - r^f_{t,s} = -\lambda^p_s \frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}. \tag{A.4}$$

Likewise, applying Eq. (20) for type i investors, we obtain:

$$\omega^i_{t,s} - 1 = \left(\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}} - 1\right)\frac{Q_{t,s'}}{Q_{t,s'} - Q_{t,s}}. \tag{A.5}$$

Substituting Eqs. ($A.4$) and ($A.5$) into Eq. ($A.3$), we obtain Eq. (22) in the main text.

A.2. Omitted derivations in Section 3

A.2.1. Equilibrium in the recession and the recovery states

As we describe in the main text, for the rest of the analysis we often simplify the notation by dropping the subscript o from optimists' wealth share:

$$\alpha_{t,s} \equiv \alpha^o_{t,s}.$$

Pessimists' wealth share is the complement of this expression, $\alpha^p_{t,s} = 1 - \alpha_{t,s}$.

We next present the details of our characterization of equilibrium for the recession and recovery states, $s \in \{2, 3\}$. We assume the following:

Assumption A1. $\delta(0) - \left(\rho + \lambda^i_2\right) < g_2 < \delta(\eta^*) - \rho < g_3$.

With this assumption, we conjecture an equilibrium in which the recovery state $s = 3$ features positive interest rates, efficient asset prices, and efficient factor utilization, $r^f_{t,3} > 0, Q_{t,3} = Q^*$ and $\eta_{t,3} = \eta^*$. The recession state $s = 2$ features an interest rate of zero, lower asset prices, and inefficient factor utilization, $r^f_{t,2} = 0, Q_{t,2} < Q^*$ and $\eta_{t,2} < \eta^*$. We will show that the equilibrium price in the recession state can be represented as a strictly increasing function of optimists' wealth share: $Q_{t,2} = Q_2(\alpha_{t,2})$ where $Q_2(\cdot)$ is a strictly increasing function.

Note that for $s \in \{2,3\}$ the leverage limit doesn't bind. Therefore, Eq. (23) applies. Combining Eqs. (21) and (23), we obtain:

$$r_{t,s} - r_{t,s}^f + \overline{\lambda}_{t,s} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} = 0. \tag{A.6}$$

In particular, the risk premium is determined by the weighted-average belief, $\overline{\lambda}_{t,s}$.

Equilibrium in the recovery state $s = 3$. In the recovery state, there is no speculation since $\lambda_3^i = 0$ for each i. Substituting this transition probability into Eq. ($A.6$), we find that the risk premium is zero, $r_{t,3} - r_{t,3}^f = 0$. After substituting for the market return from Eq. (18), and using $\dot{Q}_{t,3} = 0$ (since $Q_{t,3} = Q^*$ is constant), we obtain:

$$r_{t,3}^f = \rho + g_3 - \delta(\eta^*) > 0. \tag{A.7}$$

The inequality follows from Assumption A1. Hence, in the recovery state, the interest rate is constant and strictly positive and the equilibrium asset price and factor utilization levels are efficient.

Equilibrium in the recession state $s = 2$. In this state, there is some speculation since investors have heterogeneous beliefs, $\lambda_2^o > \lambda_2^p$ [cf. Assumption 1]. Substituting Eq. (18) into Eq. ($A.6$) and using the conjecture $Q_{t,3} = Q^*$, we obtain Eq. (25) in the main text. Substituting the conjecture $r_{t,2}^f = 0$, we further obtain:

$$\rho + g_2 - \delta\left(\frac{Q_{t,2}}{Q^*}\eta^*\right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_{t,2}}{Q^*}\right) = 0. \tag{A.8}$$

Next consider the extreme cases $\alpha_{t,2} \in \{0,1\}$. These cases are the same as if there is a single belief type $i \in \{o,p\}$. In particular, since there is no speculation, the price is constant within the state, that is: $Q_{t,2} \equiv Q_2^i$ and thus $\dot{Q}_{t,2} = 0$. Therefore, Eq. ($A.8$) can be written as

$$\rho + g_2 - \delta\left(\frac{Q_2^i}{Q^*}\eta^*\right) + \lambda_2^i\left(1 - \frac{Q_2^i}{Q^*}\right) = 0.$$

Under Assumption A1, there exists a solution that satisfies $Q_2^i \in (0, Q^*)$. This describes the equilibrium price in the recession state if all investors share type i investors' beliefs. Using $\lambda_2^o > \lambda_2^p$ (Assumption 1), it is easy to check that $Q_2^o > Q_2^p$. In particular, the price is greater under optimists' beliefs than under pessimists' beliefs.

Next consider the intermediate cases, $\alpha_{t,2} \in (0,1)$. In this case we combine Eq. ($A.8$) with Eq. (24) for state $s = 2$ to obtain a system of differential equations for $(\alpha_{t,2}, Q_{t,2})$:

$$\begin{aligned} &\rho + g_2 - \delta\left(\frac{Q_{t,2}}{Q^*}\eta^*\right) + \frac{\dot{Q}_{t,2}}{Q_{t,2}} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_{t,2}}{Q^*}\right) = 0, \\ &\dot{\alpha}_{t,2} = -\alpha_{t,2}(1 - \alpha_{t,2})\Delta\lambda_2^o. \end{aligned} \tag{A.9}$$

This is similar to the differential equation system for the recession state in Caballero and Simsek (2017). Following similar steps, we show that the system is saddle path stable: for any $\alpha_{t,2}$, there exists a unique equilibrium price level $Q_{t,2} \in [Q_2^p, Q_2^o)$ such that the solution satisfies $\lim_{t\to\infty} \alpha_{t,2} = 0$ and $\lim_{t\to\infty} Q_{t,2} = Q_2^p$. Since the system is stationary, the solution can be written as a function of optimists'

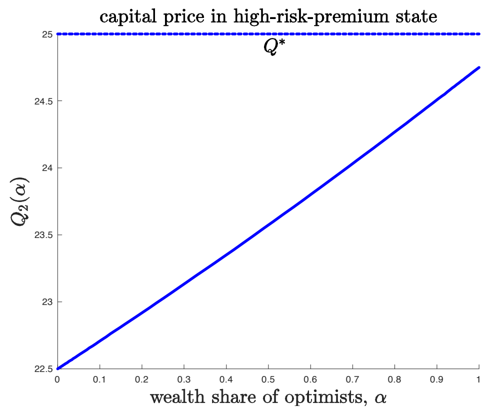

Figure 9: Equilibrium price of capital in the recession state $s = 2$. The dotted line illustrates the frictionless price level, Q^*.

wealth share, $Q_{t,2} = Q_2(\alpha)$. In Caballero and Simsek (2017), we show that $Q_2(\alpha)$ is strictly increasing in α. Since $Q_2^p < Q_2^o < Q^*$, this establishes Eq. (26) in the main text.

For a numerical solution, we convert the differential equation in (A.9) into a differential equation in α-domain. In particular, differentiating $Q_{t,2} = Q_2(\alpha_{t,2})$ with respect to time, we obtain:

$$\dot{Q}_{t,2} = Q_2'(\alpha_{t,2})\,\dot{\alpha}_{t,2}.$$

Combining this with Eq. (A.9), we obtain:

$$\frac{Q_2'(\alpha)}{Q_2(\alpha)} = \frac{1}{\alpha(1-\alpha)\Delta\lambda_2^o}\left(\rho + g_2 - \delta\left(\frac{Q_2(\alpha)}{Q^*}\eta^*\right) + \overline{\lambda}_2(\alpha)\left(1 - \frac{Q_2(\alpha)}{Q^*}\right)\right).$$

The equilibrium price function is the solution to this system subject to the boundary conditions $Q_2(0) = Q_2^p$ and $Q_2(1) = Q_2^o$. Figure 9 illustrates the solution for a particular parameterization.

A.2.2. Value functions in equilibrium

We next characterize investors' equilibrium expected values and derive the gap value that we use in the main text. Let the superscript $b \in \{o, p, pl\}$ denote the belief corresponding to optimists, pessimists, or the planner. Let $i \in \{o, p\}$ denote type i investors. We let $V_{t,s}^{i,b}\left(a_{t,s}^i\right)$ denote type i investors' expected value when she has wealth $a_{t,s}^i$, evaluated according to type b belief. In view of log utility, we conjecture the following version of Eq. (A.2):

$$V_{t,s}^{i,b}\left(a_{t,s}^i\right) = \frac{\log\left(a_{t,s}^i / Q_{t,s}\right)}{\rho} + v_{t,s}^{i,b}. \tag{A.10}$$

Note that this function implies $\frac{\partial V^{i,b}_{t,s}}{\partial a} = \frac{1}{\rho a^i_{t,s}}$. Using this expression as well as $c^i_{t,s} = \rho a^i_{t,s}$, we obtain the following version of the HJB equation (A.1):

$$\rho V^{i,b}_{t,s}\left(a^i_{t,s}\right) - \frac{\partial V^i_{t,s}\left(a^{i,b}_{t,s}\right)}{\partial t} = \log \rho a^i_{t,s} + \frac{1}{\rho}\left(r^f_{t,s} + \omega^i_{t,s}\left(r_{t,s} - r^f_{t,s}\right) - \rho\right) + \lambda^b_s\left(V^i_{t,s'}\left(a^i_{t,s}\left(1 + \omega \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right)\right) - V^i_{t,s}\left(a^i_{t,s}\right)\right). \tag{A.11}$$

Note that we evaluate the value function along the equilibrium path and according to transition probability λ^b_s.

Substituting Eq. (A.10) into Eq. (A.11), we obtain a differential equation for the normalized value:

$$\rho v^{i,b}_{t,s} - \frac{\partial v^{i,b}_{t,s}}{\partial t} = \log \rho + \log Q_{t,s} + \frac{1}{\rho}\left(r_{t,s} - \rho - \frac{\dot{Q}_{t,s}}{Q_{t,s}} + \left(\omega^i_{t,s} - 1\right)\left(r_{t,s} - r^f_{t,s}\right) + \lambda^b_s \log\left(1 + \omega \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}\right)\right) + \lambda^b_s\left(v^{i,b}_{t,s'} - v^{i,b}_{t,s}\right).$$

To simplify this expression, we substitute $r_{t,s} = \rho + \frac{\dot{Q}_{t,s}}{Q_{t,s}} + g_s - \delta\left(\frac{Q_{t,s}}{Q^*}\eta^*\right)$ using Eq. (18). We also substitute for $\left(\omega^i_{t,s} - 1\right)\left(r_{t,s} - r^f_{t,s}\right) = \frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}}$ from Eq. (A.3). Finally, we substitute for $1 + \omega\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}} = \frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}}$ using Eq. (20). After these substitutions, we obtain:

$$\rho v^{i,b}_{t,s} - \frac{\partial v^{i,b}_{t,s}}{\partial t} = \log \rho + \log Q_{t,s} + \frac{1}{\rho}\left(g_s - \delta\left(\frac{Q_{t,s}}{Q^*}\eta^*\right) + \frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} + \lambda^b_s \log\left(\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}}\right)\right) + \lambda^b_s\left(v^{i,b}_{t,s'} - v^{i,b}_{t,s}\right). \tag{A.12}$$

We have thus characterized the normalized value function, $v^{i,b}_{t,s}$, as a solution to the differential equation in (A.12). This equation applies for any beliefs $b \in \{o, p, pl\}$, including investors' own beliefs $b = i$, and it applies regardless of whether the leverage limit binds. The terms that feature $Q_{t,s}$ capture potential welfare losses due to inefficient factor utilization. The term g_s captures the welfare effect of expected growth. The term $\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} + \lambda^b_s \log\left(\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}}\right)$ captures the welfare effect of speculation that reshuffles investors' wealth shares across states.

As we describe in the main text, we decompose the normalized value into two components [cf. (27)]:

$$v^{i,b}_{t,s} = v^{i*,b}_{t,s} + w^{i,b}_{t,s},$$

Here, $v^{i*,b}_{t,s}$ is *the frictionless value function*, which is found by solving Eq. (A.12) with $Q_{t,s} = Q^*$ for each t, s. This captures all determinants of welfare except for suboptimal factor utilization (including the benefits/costs from speculation). The residual, $w^b_{t,s}$, corresponds to *the gap value function*. This captures the welfare losses due to suboptimal factor utilization evaluated according to investors' preferences (and type b beliefs).

To further characterize the gap value, note that $v^{i,b}_{t,s}$ and $v^{i*,b}_{t,s}$ both solve Eq. (A.12) with $Q_{t,s}$ and $Q_{t,s} = Q^*$, respectively. Taking the difference of these equations, and using $w^{i,b}_{t,s} = v^{i,b}_{t,s} - v^{i*,b}_{t,s}$, we obtain

Eq. (28) in the main text, which we replicate for ease of exposition:

$$\begin{aligned}
\rho w_{t,s}^{b}-\frac{\partial w_{t,s}^{b}}{\partial t} &= W\left(Q_{t,s}\right)+\lambda_{s}^{b}\left(w_{t,s'}^{b}-w_{t,s}^{b}\right), \\
\text{where } W\left(Q_{t,s}\right) &= \log\frac{Q_{t,s}}{Q^{*}}-\frac{1}{\rho}\left(\delta\left(\frac{Q_{t,s}}{Q^{*}}\eta^{*}\right)-\delta\left(\eta^{*}\right)\right).
\end{aligned}$$

This implies that the gap value depends on an investor's beliefs but not her identity, $w_{t,s}^{b}\equiv w_{t,s}^{i,b}$.

Integrating Eq. (28) forward, we obtain:

$$w_{t,s}^{b}=\int_{t}^{\infty}e^{-\left(\rho+\lambda_{s}^{b}\right)\left(\tilde{t}-t\right)}\left(W\left(Q_{\tilde{t},s}\right)+\lambda_{s}^{b}w_{\tilde{t},s'}^{b}\right)d\tilde{t}. \tag{A.13}$$

Hence, the gap value captures an appropriately discounted present value of instantaneous welfare gaps. Note that $W\left(Q_{t,s}\right)$ is a strictly concave function maximized at $Q_{t,s}=Q^{*}$. Therefore, Eq. (A.13) also implies $w_{t,s}^{b}\leq 0$ for each t,s.

A.2.3. Gap value in recession

Next consider the gap value in the recession state $s=2$. Since the model is stationary, we conjecture that the gap value can be written as a function of optimists' wealth share,

$$w_{t,2}^{b}=w_{2}^{b}\left(\alpha_{t,s}\right),$$

for some function $w_{2}^{b}\left(\cdot\right)$. Differentiating this expression, we have:

$$\begin{aligned}
\frac{\partial w_{t,s}^{b}}{\partial t} &= \frac{dw_{2}^{b}\left(\alpha_{t,s}\right)}{d\alpha}\dot{\alpha}_{t,s} \\
&= -\frac{dw_{2}^{b}\left(\alpha_{t,s}\right)}{d\alpha}\alpha_{t,2}\left(1-\alpha_{t,2}\right)\Delta\lambda_{2}^{o}.
\end{aligned}$$

Note that $w_{t,3}^{b}=0$ since $Q_{t,3}=Q^{*}$. Finally, recall that we have $Q_{t,2}=Q_{2}\left(\alpha\right)<Q^{*}$, where $Q_{2}\left(\alpha\right)$ is a strictly increasing function. Substituting these expressions into Eq. (28) for state $s=2$, we characterize the gap value as the solution to a differential equation in α-domain:

$$\left(\rho+\lambda_{2}^{b}\right)w_{2}^{b}\left(\alpha\right)+\frac{dw_{2}^{b}\left(\alpha\right)}{d\alpha}\alpha\left(1-\alpha\right)\Delta\lambda_{2}^{o}=W\left(Q_{2}\left(\alpha\right)\right).$$

We analyze the solution to this differential equation in Caballero and Simsek (2017). In particular, since $W\left(Q_{2}\left(\alpha\right)\right)$ is strictly increasing in α (since $Q_{2}\left(\alpha\right)<Q^{*}$), $w_{2}^{b}\left(\alpha\right)$ is also strictly increasing in α. Using the integral expression in (A.13), we also have $w_{2}^{b}\left(\alpha\right)<0$ for each α. This establishes Eq. (29) in the main text.

A.3. Omitted derivations in Section 4

We first characterize the equilibrium for a given leverage limit function, $\overline{\omega}_{1}\left(\cdot\right)$. We then prove Proposition 1, which establishes the comparative statics of tightening the leverage limit (for given α).

To characterize the equilibrium, we assume the parameters satisfy:

Assumption A2. $r_{1}^{f,p}\equiv\rho+g_{1}-\delta\left(\eta^{*}\right)-\lambda_{1}^{p}\left(\frac{Q^{*}}{Q_{2}^{p}}-1\right)>0.$

Here, $Q_2^p = Q_2(0) < Q^*$ denotes the asset price in the recession state when pessimists dominate the economy. Assumption A2 ensures that the boom features a positive interest rate even if pessimists dominate. Under this assumption, we conjecture an equilibrium in which the interest rate is positive, $r_{t,1}^f > 0$, and the asset price is at its efficient level, $Q_{t,1} = Q^*$. We also conjecture that the equilibrium outcomes can be described as a function of optimists' wealth share, $\alpha_{t,1}$ (as well as the leverage limit function, $\overline{\omega}_1(\cdot)$). In particular, optimists' wealth share after transition can be written as $\alpha_{t,2} = \alpha_2(\alpha_{t,1}, \overline{\omega}_1)$ (and pessimists' wealth share is the residual, $\alpha_{t,2}^p = 1 - \alpha_{t,2}$).

First consider the corner cases $\alpha_{t,1} = 0$ and $\alpha_{t,1} = 1$. Equivalently, $\alpha_{t,1}^i = 1$ for some belief type i. Using Eq. (21), which holds as equality for type i investors, we obtain:

$$r_1^{f,i} = \rho + g_1 - \delta(\eta^*) - \lambda_1^i \left(\frac{Q^*}{Q_2^i} - 1 \right). \tag{A.14}$$

Under Assumption A2, there exists a solution that satisfies $r_1^{f,i} > 0$ for each $i \in \{o, p\}$. Since $\lambda_1^o < \lambda_1^p$, we also have $r_1^{f,o} > r_1^{f,p}$: the equilibrium interest rate is greater when optimists dominate the economy.

Next consider the intermediate cases, $\alpha_{t,1} \in (0,1)$. Most of the analysis is in the main text. The remaining step is to show that, when $\overline{\omega}_1(\alpha) \leq \omega_1^o(\alpha, \infty)$ (when the leverage limit binds) Eq. (34) has a unique solution that satisfies $\alpha_2(\alpha, \overline{\omega}_1) \geq \alpha_2(\alpha, \infty)$. This result follows from Lemma 1 below, which we use in subsequent sections. The lemma applies under the following regularity conditions:

Assumption A3. $Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1-\alpha_2}$ for $\alpha_2 \in (0,1)$; and $Q_2\left(\frac{\alpha \lambda_1^o}{\overline{\lambda}_1(\alpha)}\right) > Q^* \alpha \left(1 - \frac{\lambda_1^o}{\lambda_1^p}\right)$ for $\alpha \in (0,1)$.

These conditions concern the price function in the recession. They are mild, and we can verify that numerical solutions do not violate these conditions. They are also sufficient conditions, i.e., they can be relaxed further. The first part says that the slope of the price function is not too large. Since $Q_2(1) = Q_2^o < Q^*$, this condition will always hold if $Q_2(\alpha_2)$ is a linear function. Therefore, it holds as long as $Q_2(\alpha_2)$ does not deviate from linearity too much. The second part requires that either the price decline after transition to the recession is not too large, or the extent of speculation during the boom is not too large. For instance, when $\alpha = 1$, the requirement is $Q_2^o > Q^*\left(1 - \frac{\lambda_1^o}{\lambda_1^p}\right)$. This holds if Q_2^o is close to Q^* or if λ_1^o is not substantially smaller than λ_1^p.

Lemma 1. *Consider the following function:*

$$f(\alpha_2; \alpha, \overline{\omega}_1) = 1 - \frac{\alpha_2}{\alpha} - (\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right],$$

where $\alpha, \overline{\omega}_1$ are parameters such that $\alpha \in (0,1), \overline{\omega}_1 \leq \omega_1^o(\alpha, \infty)$. Under Assumption A3, $f(\alpha_2) = 0$ has a unique solution that satisfies $\alpha_2 \in [\alpha_2(\alpha, \infty), \alpha)$.

Proof. We first show that there exists a solution that lies in the desired interval. We have

$$\begin{aligned} f(\alpha_2(\alpha, \infty)) &= 1 - \frac{\alpha_2(\alpha,\infty)}{\alpha} - (\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\infty))} - 1\right] \\ &\geq 1 - \frac{\alpha_2(\alpha,\infty)}{\alpha} - (\omega_1^o(\alpha,\infty) - 1)\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\infty))} - 1\right] = 0. \end{aligned}$$

Here, the inequality in the second line follows since $\overline{\omega}_1 \leq \omega_1^o(\alpha, \infty)$ and $Q_2(\alpha_2(\alpha, \infty)) < Q^*$, and the equality follows from the definition of $\omega_1^o(\alpha,\infty)$. We also have

$$f(\alpha) = -(\overline{\omega}_1 - 1)\left[\frac{Q^*}{Q_2(\alpha_2)} - 1\right] < 0.$$

It follows that there exists a solution in $[\alpha_2(\alpha,\infty), \alpha)$.

We next show that the derivative of f is strictly negative at each zero of f:

$$f'(\alpha_2) < 0 \text{ for each } \alpha_2 \in [\alpha_2(\alpha,\infty), \alpha) \text{ and } f(\alpha_2) = 0. \tag{A.15}$$

This establishes that f has a unique zero in the desired interval. To establish this claim, we first evaluate the derivative

$$f'(\alpha_2) = -\frac{1}{\alpha} + (\overline{\omega}_1 - 1)\frac{Q^*}{(Q_2(\alpha_2))^2}Q_2'(\alpha_2).$$

Hence, $f'(\alpha_2) < 0$ as long as

$$\alpha(\overline{\omega}_1 - 1)\frac{Q^*}{Q_2(\alpha_2)}\frac{Q_2'(\alpha_2)}{Q_2(\alpha_2)} < 1.$$

Note that we require this to hold when $f(\alpha_2) = 0$. This implies

$$\alpha(\overline{\omega}_1 - 1)\frac{Q^*}{Q_2(\alpha_2)} = (\alpha - \alpha_2)\frac{Q^*}{Q^* - Q_2(\alpha_2)}.$$

Combining the last two displayed equations, we need to show

$$Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1 - \alpha_2}\frac{1 - \alpha_2}{\alpha - \alpha_2}\frac{Q_2(\alpha_2)}{Q^*}. \tag{A.16}$$

Using the first part of Assumption A3, we have

$$Q_2'(\alpha_2) < \frac{Q^* - Q_2(\alpha_2)}{1 - \alpha_2}. \tag{A.17}$$

Using the second part of Assumption A3, we also have

$$1 \leq \frac{1 - \alpha_2(\alpha,\infty)}{\alpha - \alpha_2(\alpha,\infty)}\frac{Q_2(\alpha_2(\alpha,\infty))}{Q^*} \leq \frac{1 - \alpha_2}{\alpha - \alpha_2}\frac{Q_2(\alpha_2)}{Q^*}. \tag{A.18}$$

Here, the first inequality follows from Assumption A3 since $\frac{1-\alpha_2(\alpha,\infty)}{\alpha-\alpha_2(\alpha,\infty)} = \frac{\lambda_1^p}{\alpha\left(\lambda_1^p - \lambda_1^o\right)}$ after substituting $\alpha_2(\alpha,0) = \frac{\alpha\lambda_1^o}{\lambda_1(\alpha)}$ [cf. Eq. (31)]. The second inequality follows since $\alpha_2(\alpha,\infty) \leq \alpha_2$ implies $\frac{1-\alpha_2(\alpha,\infty)}{\alpha-\alpha_2(\alpha,\infty)} \leq \frac{1-\alpha_2}{\alpha-\alpha_2}$ and $Q_2(\alpha_2(\alpha,\infty)) \leq Q_2(\alpha_2)$. Combining Eqs. ($A.17$) and ($A.18$) establishes Eq. ($A.16$). This in turn establishes Eq. ($A.15$) and shows that there is a unique solution. $\square$

Proof of Proposition 1. Recall that optimists' wealth share after transition corresponds to the zero of the function defined in Lemma 1. Next consider how the solution (characterized in the proof of the lemma) changes with $\overline{\omega}_1$. Implicitly differentiating the equation $f(\alpha_2;\alpha,\overline{\omega}_1) = 0$ with respect to $\overline{\omega}_1$, we obtain:

$$\frac{d\alpha_2}{d\overline{\omega}_1} = \frac{\frac{Q^*}{Q_2(\alpha_2)} - 1}{f'(\alpha_2)} < 0.$$

Here, the inequality follows since $\frac{Q^*}{Q_2(\alpha_2)} - 1 > 0$ and $f'(\alpha_2) < 0$ [cf. Eq. ($A.15$)]. It follows that the solution is strictly decreasing in $\overline{\omega}_1$, that is, $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$. In particular, decreasing the leverage limit increases optimists' wealth share after transition.

To establish the last part, note that Eq. (35) describes optimists' growth absent transition, $\dot{\alpha}_{t,1}/\alpha_{t,1}$, as a *decreasing* function of $\alpha_{t,2}$ (given the parameters and $\alpha_{t,1}$). Combining this observation with $\frac{d\alpha_2(\alpha,\overline{\omega}_1)}{d\overline{\omega}_1(\alpha)} < 0$, we also find $\frac{d(\dot{\alpha}_{t,1}/\alpha_{t,1})}{d\overline{\omega}_1(\alpha)} < 0$. Hence, decreasing the leverage limit slows down the growth of optimists' wealth share absent transition, completing the proof. □

A.4. Omitted derivations in Section 5

Proof of Proposition 2. Recall that Eq. (23) applies for an arbitrary specification of monetary policy as long as leverage constraints do not bind for either type. When $\overline{\omega}_1 = \infty$, constraints do not bind in state 1. Applying Eq. (23), the evolution of optimists' wealth share is given by (31). In particular, monetary policy does not influence the evolution of optimists' wealth share.

Next note that, using Eq. (*A*.13), we can write the planner's gap value as

$$w_1^{pl}(\alpha_{0,1}) = \int_0^\infty e^{-(\rho+\lambda_1^{pl})t}\left(W(Q_{t,1}) + \lambda_1^{pl} w_{t,2}^b(\alpha_{t,2})\right)dt.$$

Here, $\alpha_{t,2}$ denotes optimists' wealth share in the recession state if the economy switches to recession at time t. Monetary policy does not affect the path $\{\alpha_{t,2}\}$. Therefore, the previous expression is maximized when $W(Q_{t,1})$ is maximized. This happens when the planner follows the conventional output-stabilization policy and sets $Q_{t,1} = Q^*$. It follows that prudential policy can only lower the gap value function, $w_1^{pl}(\alpha,\infty,Q_1) \leq w_1^{pl}(\alpha,\infty)$, completing the proof. □

Proof of Proposition 3. First consider the effect of the leverage limit, $\tilde{\omega}_1$. Since $\tilde{\omega}_1(\alpha) < \omega_1(\alpha,\infty)$, optimists' wealth share, $\alpha_2(\alpha,\tilde{\omega}_1)$, is characterized as the unique solution to the following equation (see Appendix A.3):

$$\frac{\alpha_2(\alpha,\tilde{\omega}_1)}{\alpha} = 1 - (\tilde{\omega}_1(\alpha) - 1)\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\tilde{\omega}_1))} - 1\right]. \tag{A.19}$$

We will show (constructively) that there exists a PMP that replicates the wealth share. Let $\tilde{\alpha}_2 = \alpha_2\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right)$ denote optimists' wealth share after transition with PMP. In the conjectured equilibrium, optimists' leverage limit binds (since $\tilde{\alpha}_2 = \alpha_2(\alpha,\tilde{\omega}_1) > \alpha_2(\alpha,\infty)$). Therefore, optimists' wealth share is the solution to

$$\frac{\tilde{\alpha}_2}{\alpha} = 1 - (\overline{\omega}_1(\alpha) - 1)\left[\frac{\tilde{Q}_1(\alpha)}{Q_2(\tilde{\alpha}_2)} - 1\right]. \tag{A.20}$$

We next claim that, for appropriately chosen $\tilde{Q}_1(\alpha)$, this equation holds for $\tilde{\alpha}_2 = \alpha_2(\alpha,\tilde{\omega}_1)$.

To this end, let $\tilde{Q}_1(\alpha)$ be such that Eq. (37) holds. After rearranging this expression, we can solve for $\tilde{Q}_1(\alpha)$ in closed form:

$$\tilde{Q}_1(\alpha) = Q_2(\alpha_2(\alpha,\tilde{\omega}_1))\left(1 + \frac{\tilde{\omega}_1(\alpha) - 1}{\overline{\omega}_1(\alpha) - 1}\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\tilde{\omega}_1))} - 1\right]\right). \tag{A.21}$$

Since $\tilde{\omega}_1(\alpha) < \overline{\omega}_1(\alpha)$, it is easy to check that $\tilde{Q}_1(\alpha) < Q^*$. Since $\tilde{\omega}_1(\alpha) > 1$, we also have $\tilde{Q}_1(\alpha) > Q_2(\alpha_2(\alpha,\tilde{\omega}_1))$. In particular, there exists a unique $\tilde{Q}_1(\alpha) \in (Q_2(\alpha_2(\alpha,\tilde{\omega}_1)), Q^*)$ that satisfies Eq. (37).

We next substitute Eq. (37) into Eq. (*A*.19), which proves our claim that Eq. (*A*.20) holds with $\tilde{\alpha}_2 = \alpha_2(\alpha,\tilde{\omega}_1)$. We can also check that (under Assumption A3) this equation has a unique solution.

This proves $\alpha_2\left(\alpha, \overline{\omega}_1, \tilde{Q}_1\right) = \alpha_2\left(\alpha, \tilde{\omega}_1\right)$. Note that Eq. (A.21) implies $\frac{\partial \tilde{Q}_1(\alpha)}{\partial \tilde{\omega}_1(\alpha)} > 0$, which completes the proof of the first part of the proposition .

Next consider the interest rate corresponding to PMP. Since the policy applies only at an infinitesimal instant, it does not affect the price drift, $\dot{Q}_{t,1} = 0$. In particular, the instantaneous return to capital is given by $\tilde{r}_1 = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right)$ [cf. Eq. (18)]. Combining this with Eq. (21) for pessimists, we obtain the following analogue of Eq. (30):

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) - \lambda_1^p \frac{1-\alpha}{1-\alpha_2(\alpha,\tilde{\omega}_1)}\left(\frac{\tilde{Q}_1(\alpha)}{Q_2(\alpha_2(\alpha,\tilde{\omega}_1))} - 1\right).$$

Using Eq. (A.20) to substitute for the price decline, we can rewrite this as

$$\tilde{r}_1^f = \rho + g_1 - \delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) - \lambda_1^p \frac{1-\alpha}{\alpha}\frac{\alpha - \alpha_2(\alpha,\tilde{\omega}_1)}{1-\alpha_2(\alpha,\tilde{\omega}_1)}\frac{1}{\overline{\omega}_1(\alpha) - 1}. \tag{A.22}$$

Absent prudential policy, the interest rate is characterized by Eq. (30). After substituting for the price decline from (20), we can rewrite this expression as

$$r_1^f(\alpha,\overline{\omega}_1) = \rho + g_1 - \delta(\eta^*) - \lambda_1^p \frac{1-\alpha}{\alpha}\frac{\alpha_2(\alpha,\overline{\omega}_1) - \alpha}{1-\alpha_2(\alpha,\overline{\omega}_1)}\frac{1}{\omega_1(\alpha,\overline{\omega}_1) - 1}. \tag{A.23}$$

Here, $\omega_1(\alpha,\overline{\omega}_1)$ denotes the equilibrium leverage ratio.

Next note that $\delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right) < \delta(\eta^*)$ since $\tilde{Q}_1(\alpha) < Q^*$. Note also that $\frac{\alpha-\alpha_2(\alpha,\tilde{\omega}_1)}{1-\alpha_2(\alpha,\tilde{\omega}_1)} < \frac{\alpha-\alpha_2(\alpha,\overline{\omega}_1)}{1-\alpha_2(\alpha,\overline{\omega}_1)}$ since $\alpha_2(\alpha,\tilde{\omega}_1) > \alpha_2(\alpha,\overline{\omega}_1)$. Finally, note that $\frac{1}{\overline{\omega}_1(\alpha)-1} \leq \frac{1}{\omega_1(\alpha,\overline{\omega}_1)-1}$ since $\omega_1(\alpha,\overline{\omega}_1) \leq \overline{\omega}_1(\alpha)$. Combining these observations with Eqs. (A.22) and (A.23) proves that $\tilde{r}_1^f = r_1^f\left(\alpha,\overline{\omega}_1,\tilde{Q}_1\right) > r_1^f(\alpha,\overline{\omega}_1)$: PMP raises the interest rate.

Finally, consider how raising the leverage limit $\tilde{\omega}_1(\alpha)$ affects the interest rate with PMP. Since raising the leverage limit increases $\tilde{Q}_1(\alpha)$, it also increases the effective depreciation rate, $\delta\left(\frac{\tilde{Q}_1(\alpha)}{Q^*}\eta^*\right)$. Since raising the leverage limit reduces $\alpha_2(\alpha,\tilde{\omega}_1)$, it also increases the term $\frac{\alpha-\alpha_2(\alpha,\tilde{\omega}_1)}{1-\alpha_2(\alpha,\tilde{\omega}_1)}$. Combining these observations with (A.22) proves that raising the leverage limit decreases $\tilde{r}_1$, that is: $\frac{\partial r_1^f(\alpha,\overline{\omega}_1,\tilde{Q}_1)}{\partial \tilde{\omega}_1(\alpha)} < 0$. In particular, lowering the effective leverage limit $\tilde{\omega}_1(\alpha)$ requires a higher interest rate, completing the proof. □

Proof of Proposition 4. We have the following closed-form solution for the price function:

$$Q_1\left(\alpha,\tilde{l}\right) = \begin{cases} Q^* & \text{if } \omega_1\left(\alpha,\tilde{l}\right) < \overline{\omega}_1\left(\alpha,\tilde{l}\right) \\ Q_2\left(\alpha_2\left(\alpha,\tilde{l}\right)\right)\left(1 + \frac{\overline{\omega}_1(\alpha,\tilde{l})-1}{\overline{\omega}_1(\alpha,l)-1}\left[\frac{Q^*}{Q_2(\alpha_2(\alpha,\tilde{l}))} - 1\right]\right) < Q & \text{if } \omega_1\left(\alpha,\tilde{l}\right) = \overline{\omega}_1\left(\alpha,\tilde{l}\right) \end{cases}. \tag{A.24}$$

Here, the first line corresponds to the case in which the leverage limit does not bind under $\tilde{l}$. In this case, the monetary authority does not use PMP. The second line corresponds to the case in which the leverage limit binds. In this case, the monetary authority uses PMP. Moreover, using Eq. (A.21) we have a closed-form solution for the asset price level.

One difference from Proposition 3 concerns the characterization of the interest rate. Since the policy is applied dynamically, the price drift, $\dot{Q}_{t,1}$, is not necessarily zero, which affects the level of the interest rate. To characterize this effect, note that:

$$
\begin{aligned}
\dot{Q}_{t,1} &= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\dot{\alpha}_{t,1} \\
&= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\lambda_1^p\alpha_{t,1}\frac{1-\alpha_{t,1}}{1-\alpha_{t,2}}\left(\alpha_{t,1}-\alpha_{t,2}\right) \\
&= \frac{\partial Q_1\left(\alpha,\tilde{l}\right)}{\partial_+\alpha}\lambda_1^p\alpha\left(1-\alpha\right)\frac{\alpha-\alpha_2\left(\alpha,\tilde{l}\right)}{1-\alpha_2\left(\alpha,\tilde{l}\right)}.
\end{aligned}
\tag{A.25}
$$

Here, the second line substitutes the evolution of optimists' wealth share from Eq. (22) and the third line substitutes $\alpha_{t,1}=\alpha$ and $\alpha_{t,2}=\alpha_2\left(\alpha,\tilde{l}\right)$. The expression $\frac{\partial Q_1(\alpha,\tilde{l})}{\partial_+\alpha}$ corresponds to the right-derivative of the function characterized in (A.24).[10] We can check that the right-derivative, $\frac{\partial Q_1(\alpha,\tilde{l})}{\partial_+\alpha}$, is continuous in $\tilde{l}$ and equal to 0 when $\tilde{l}=l$ (because $Q_1\left(\alpha,l\right)=Q^*$ for each α). Consequently, when viewed as a function of $\tilde{l}$, the price drift, $\dot{Q}_{t,1}$, is also continuous in $\tilde{l}$ and equal to 0 when $\tilde{l}=l$.

Next note that, following similar steps as in the proof of Proposition 4, the interest rate in this case can be written as

$$
\tilde{r}_1^f=\rho+g_1+\dot{Q}_{t,1}-\delta\left(\frac{Q_1\left(\alpha,\tilde{l}\right)}{Q^*}\eta^*\right)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2\left(\alpha,\tilde{l}\right)}\left(\frac{Q_1\left(\alpha,\tilde{l}\right)}{Q_2\left(\alpha_2\left(\alpha,\tilde{l}\right)\right)}-1\right),
$$

where $\dot{Q}_{t,1}$ is given by Eq. (A.25). When viewed as a function of $\tilde{l}$, the interest rate $\tilde{r}_1^f$ is continuous in $\tilde{l}$, and it is equal to the benchmark interest rate $r_1^f\left(\alpha,l\right)$ when $\tilde{l}=l$. Recall that the benchmark rate is strictly positive for each $\alpha\in(0,1)$ [cf. Section 4]. Therefore, $\tilde{r}_1^f>0$ for each $\alpha\in(0,1)$ as long as $\tilde{l}$ is in a sufficiently small neighborhood of l. In particular, PMP doesn't violate the zero lower bound constraint on the interest rate.

Next consider the second part. Using Eq. (A.13), we can write the planner's gap value with policy $\tilde{l}$ as

$$
w_1^{pl}\left(\alpha_{0,1},\tilde{l}\right)=\int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\left(W\left(Q^*\right)+\lambda_1^{pl}w_{t,2}^b\left(\alpha_{t,2}\right)\right)dt. \tag{A.26}
$$

Here, $\alpha_{t,2}$ denotes optimists' wealth share if the economy transitions to recession at time t. Likewise, we write the planner's gap value with policy $Q_1\left(\alpha,\tilde{l}\right)$ as

$$
w_1^{pl}\left(\alpha_{0,1},Q_1\left(\alpha,\tilde{l}\right)\right)=\int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t}\left(W\left(Q_1\left(\alpha,\tilde{l}\right)\right)+\lambda_1^{pl}w_{t,2}^b\left(\alpha_{t,2}\right)\right)dt. \tag{A.27}
$$

Next note that, using the first part of this proposition, optimists' wealth share after transition, $\alpha_{t,2}$ (conditional on $\alpha_{t,1}$), is the same under policies $\tilde{l}$ and $Q_1\left(\cdot,\tilde{l}\right)$. Combining this observation with Eq. (22), we also find that the evolution of optimists' wealth share absent transition, $\dot{\alpha}_{t,1}/\alpha_{t,1}$, is the same under both policies. Consequently, optimists' wealth share follows an identical path under both policies. In view of this observation, after taking the difference of Eqs. (A.27) and (A.26), we obtain Eq. (40) in the main text.

[10] Note that the function is piecewise differentiable so the right-derivative always exists. The equation depends on the right-derivative (as opposed to left) because $\dot{\alpha}_{t,1}>0$, so $\alpha_{t,1}$ grows over time.

Next note that Eq. ($A.24$) implies (for a given $\alpha \in (0,1)$) that the prudential asset price level is differentiable in $\tilde{l}$ with a finite derivative. Note also that $Q_1(\alpha, l) = Q^*$. Therefore, taking the derivative of Eq. (40) with respect to $\tilde{l}$ and evaluating at $\tilde{l} = l^*$, we obtain:

$$\begin{aligned} \frac{dw_1^{pl}\left(\alpha, l, Q_1\left(\alpha, \tilde{l}\right)\right)}{d\tilde{l}}|_{\tilde{l}=l} - \frac{dw_1^{pl}\left(\alpha, \tilde{l}\right)}{d\tilde{l}}|_{\tilde{l}=l} &= \int_0^\infty e^{-\left(\rho+\lambda_1^{pl}\right)t} \frac{dW\left(Q^*\right)}{dQ_{t,1}} \frac{dQ_1\left(\alpha_{t,1}, \tilde{l}\right)}{d\tilde{l}}|_{\tilde{l}=l} dt \\ &= 0. \end{aligned}$$

Here, the first line applies the chain rule and the second line uses the observation that $\frac{dW(Q^*)}{dQ_{t,1}} = 0$ [cf. Eq. (28)]. Rearranging this expression establishes Eq. (39) and completes the proof. $\square$

A.5. Omitted derivations in Section 7

We first state and prove a generalization of Lemma 1, which implies that Eq. (48) has a unique solution (when $Q_1 = Q^*$). We then prove Propositions 5 and 6.

Lemma 2. *Consider the following function:*

$$f\left(\alpha_2; \alpha, \beta, \overline{\omega}_1\right) = 1 - \left(\overline{\omega}_1 - 1\right)\left[\frac{Q^*}{Q_2\left(\alpha_2\right)} - 1\right] - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right),$$

where $\alpha, \beta, \overline{\omega}_1$ are parameters such that $\alpha, \beta \in (0,1), \overline{\omega}_1 \leq \omega_1^o(\alpha, \beta, \infty)$. Under Assumption A3, $f(\alpha_2) = 0$ has a unique solution that satisfies $\alpha_2 \in (\alpha_2(\alpha, \infty), \alpha)$.

Proof. Following similar steps as in Lemma 1, it is easy to check that $f(\alpha_2(\alpha, \infty)) > 0$ and $f(\alpha) < 0$. This establishes that there exists a solution that lies in the desired interval, $\alpha_2 \in (\alpha_2(\alpha, \infty), \alpha)$.

We next show that the derivative of f is strictly negative at each zero of f, that is:

$$f'(\alpha_2) < 0 \text{ for each } \alpha_2 \in (\alpha_2(\alpha, \infty), \alpha) \text{ and } f(\alpha_2) = 0.$$

This establishes that f has a unique zero in the desired interval. To prove the claim, we first evaluate the derivative

$$f'(\alpha_2) = \left(\overline{\omega}_1 - 1\right)\frac{Q^*}{\left(Q_2\left(\alpha_2\right)\right)^2} Q_2'\left(\alpha_2\right) - \frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right).$$

Hence, $f'(\alpha_2) < 0$ as long as

$$\frac{\overline{\omega}_1 - 1}{Q_2\left(\alpha_2\right)}\frac{Q^*}{Q_2\left(\alpha_2\right)} Q_2'\left(\alpha_2\right) < \frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right).$$

Note that we require this to hold when $f(\alpha_2) = 0$. This implies:

$$\frac{\overline{\omega}_1 - 1}{Q_2\left(\alpha_2\right)} = \frac{1}{Q^* - Q_2\left(\alpha_2\right)}\left(1 - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)\right).$$

Combining the last two displayed equations, we need to show

$$\begin{aligned} Q_2'(\alpha_2) &< \frac{Q^* - Q_2(\alpha_2)}{1-\alpha_2}\frac{Q_2(\alpha_2)}{Q^*} g(\alpha_2, \alpha, \beta) \\ \text{where } g(\alpha_2, \alpha, \beta) &= \frac{(1-\alpha_2)\frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)}{1 - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)}. \end{aligned}$$

Note that, in the proof of Lemma 1, we already established this inequality for $\beta = 0$ (under Assumption A3). Hence, it suffices to show that $g(\alpha_2, \alpha, \beta) \geq g(\alpha_2, \alpha, 0)$. This inequality holds because,

$$g(\alpha_2, \alpha, \beta) > \frac{1-\alpha_2}{1 - \frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)} > \frac{1-\alpha_2}{1-\frac{\alpha_2}{\alpha}} = g(\alpha_2, \alpha, 0).$$

Here, the first inequality follows because $\frac{1}{1-\beta}\left(\frac{1}{\alpha} + \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right) < \frac{1}{\alpha}$. The second inequality follows because $\alpha_2 > \alpha_2(\alpha, \infty) = \frac{\alpha\lambda_1^o}{\alpha\lambda_1^o + (1-\alpha)\lambda_1^p}$ implies $\frac{\alpha_2}{1-\alpha_2} > \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha}{1-\alpha}$, which in turn implies $\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha} - \frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right) > \frac{\alpha_2}{\alpha}$. This establishes the claim and completes the proof of the lemma. □

Proof of Proposition 5. First consider the evolution of α and β absent transition to recession. Applying (22) for regulated and unregulated optimists (in state $s = 1$), we obtain:

$$\begin{aligned} \frac{d(\alpha(1-\beta))/dt}{\alpha(1-\beta)} &= \lambda_1^p \frac{1-\alpha}{1-\alpha_2}\left(1 - \frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right) \\ \frac{d(\alpha\beta)/dt}{\alpha\beta} &= \lambda_1^p \frac{1-\alpha}{1-\alpha_2}\left(1 - \frac{\alpha_2\beta_2}{\alpha\beta}\right) \end{aligned} \tag{A.28}$$

Solving these equations for $\dot{\alpha}$ and $\dot{\beta}$, we obtain Eq. (43) in the main text.

Next consider the characterization of α_2. In the main text, we established that Eq. (48) characterizes α_2. Lemma 2 implies that there exists a unique solution that satisfies $\alpha_2 \in (\alpha_2(\alpha, \infty), \alpha)$. Combining this with Eq. (43) also implies $\dot{\alpha} > 0$.

Next note that Eq. (44) characterizes β_2 conditional on α_2. Note also that $\alpha_2 > \alpha_2(\alpha, \infty) = \frac{\alpha\lambda_1^o}{\alpha\lambda_1^o + (1-\alpha)\lambda_1^p}$ implies $\frac{\alpha_2}{1-\alpha_2} > \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha}{1-\alpha}$. Combining this with Eq. (44), we obtain $\frac{\beta_2}{\beta} = \frac{\lambda_1^o}{\lambda_1^p}\frac{\alpha/(1-\alpha)}{\alpha_2/(1-\alpha_2)} < 1$. This proves $\beta_2 < \beta$. Combining this with Eq. (43) also implies $\dot{\beta} > 0$.

Next consider the comparative statics of α_2 with respect to β. Recall that α_2 is characterized as the unique solution to the equation, $f(\alpha_2; \alpha, \beta, \overline{\omega}_1) = 0$, where $f(\cdot)$ is defined in Lemma 2. Implicitly differentiating the equation with respect to β, we obtain:

$$\frac{d\alpha_2}{d\beta} = \frac{\partial f(\alpha_2; \alpha, \beta, \overline{\omega}_1)/\partial\beta}{-\partial f(\alpha_2; \alpha, \beta, \overline{\omega}_1)/\partial\alpha_2},$$

where the derivatives are evaluated at the solution. From the proof of Lemma 2, we also know that $f'(\alpha_2; \alpha, \beta, \overline{\omega}_1) < 0$ when $f(\alpha_2) = 0$. Hence, $\frac{d\alpha_2}{d\beta} < 0$ as long as $\partial f(\alpha_2; \alpha, \beta, \overline{\omega}_1)/\partial\beta < 0$. The latter inequality holds because:

$$\begin{aligned}\frac{\partial f(\alpha_2;\alpha,\beta,\overline{\omega}_1)}{\partial\beta} &= -\frac{\partial}{\partial\beta}\left(\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)\right)\\ &= -\frac{\partial}{\partial\beta}\left(\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)+\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)\\ &= -\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\right)<0.\end{aligned}$$

Here, the last inequality follows since $\frac{\alpha_2}{\alpha} > \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}$ (since $\alpha_2 > \alpha_2(\alpha,\infty)$) and $\frac{\partial}{\partial\beta}\frac{1}{1-\beta} > 0$. This proves $\frac{d\alpha_2}{d\beta} < 0$.

Next consider the limit as $\beta \to 1$. For any $\alpha_2 > \alpha_2(\alpha,\infty)$, we have

$$\begin{aligned}\lim_{\beta\to 1} f(\alpha_2;\alpha,\beta,\overline{\omega}_1) &= \lim_{\beta\to 1}\begin{bmatrix} 1-(\overline{\omega}_1-1)\left[\frac{Q^*}{Q_2(\alpha_2)}-1\right]+\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha} \\ -\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}\right)\end{bmatrix}\\ &= -\infty\end{aligned}$$

Here, the last line follows because $\frac{\alpha_2}{\alpha} > \frac{\lambda_1^o}{\lambda_1^p}\frac{1-\alpha_2}{1-\alpha}$ and $\lim_{\beta\to 1}\frac{-1}{1-\beta} = -\infty$. This also implies $\lim_{\beta\to 1}\alpha_2 = \alpha_2(\alpha,\infty)$ because α_2 is characterized as the unique solution to the equation, $f(\alpha_2;\alpha,\beta,\overline{\omega}_1) = 0$, over the range $\alpha_2 \in (\alpha_2(\alpha,\infty),\alpha)$.

Next consider the comparative statics with respect to the leverage limit, $\overline{\omega}_1 = \overline{\omega}_1(\alpha,\beta)$. Following similar steps, we obtain:

$$\begin{aligned}\frac{d\alpha_2}{d\overline{\omega}_1} &= \frac{-\partial f(\alpha_2;\alpha,\beta,\overline{\omega}_1)/\partial\overline{\omega}_1}{\partial f(\alpha_2;\alpha,\beta,\overline{\omega}_1)/\partial\alpha_2}\\ &= \frac{\frac{Q^*}{Q_2(\alpha_2)}-1}{(\overline{\omega}_1-1)\frac{Q^*}{(Q_2(\alpha_2))^2}Q_2'(\alpha_2)-\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)}<0.\end{aligned}$$

Here, the first equality evaluates the partial derivatives and the second inequality uses $\partial f(\alpha_2;\alpha,\beta,\overline{\omega}_1)/\partial\alpha_2 < 0$.

Finally, consider the sign of the cross-partial derivative $\frac{\partial}{\partial\beta}\frac{d\alpha_2}{d\overline{\omega}_1}$. We have

$$\begin{aligned}sign\left(\frac{\partial}{\partial\beta}\frac{d\alpha_2}{d\overline{\omega}_1}\right) &= sign\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1}{1-\alpha}\right)\right)\\ &= sign\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right)-\frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right)\\ &= sign\left(\frac{\partial}{\partial\beta}\frac{1}{1-\beta}\left(\frac{1}{\alpha}+\frac{\lambda_1^o}{\lambda_1^p}\frac{1}{1-\alpha}\right)\right)>0.\end{aligned}$$

This proves $\frac{\partial}{\partial\beta}\frac{d\alpha_2}{d\overline{\omega}_1} > 0$ and completes the proof. □

Proof of Proposition 6. The proof follows similar steps to Proposition 3. Using Eq. 48, α_2 corresponding to the alternative leverage limit is characterized as the unique solution to:

$$\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right) = 1-(\tilde{\omega}_1-1)\left[\frac{Q^*}{Q_2(\alpha_2)}-1\right]. \tag{A.29}$$

Likewise, α_2 corresponding to the PMP (with the current leverage limit) is characterized as the solution to:

$$\frac{1}{1-\beta}\left(\frac{\alpha_2}{\alpha}-\frac{\lambda_1^o}{\lambda_1^p}\beta\frac{1-\alpha_2}{1-\alpha}\right)=1-(\overline{\omega}_1-1)\left[\frac{\tilde{Q}_1}{Q_2(\alpha_2)}-1\right]. \tag{A.30}$$

Next note that the proof of Proposition 3 establishes that there is a unique level of $\tilde{Q}_1$ that ensures Eq. (37) holds. Let $\tilde{Q}_1$ denote this level, that is:

$$(\overline{\omega}_1-1)\left[\frac{\tilde{Q}_1}{Q_2(\alpha_2)}-1\right]=(\tilde{\omega}_1-1)\left[\frac{Q^*}{Q_2(\alpha_2)}-1\right]. \tag{A.31}$$

Substituting $\tilde{Q}_1$ into Eq. ($A.30$) ensures that this equation is the same as Eq. ($A.29$). Therefore, the solutions are the same. Hence, there exists a PMP that replicates α_2 that results from the alternative leverage limit. Recall also that β_2 is characterized by Eq. (44) conditional on α_2. Thus, the same PMP also replicates β_2 that results from the alternative leverage limit. Note also that Eq. ($A.31$) implies $\frac{\partial\tilde{Q}_1}{\partial\tilde{\omega}_1}>0$. This completes the proof of the first part.

Next consider the interest rate corresponding to PMP. Note that Eq. (21) continues to hold as equality for pessimists. This implies that the interest rate is given by:

$$\tilde{r}_1^f=\rho+g_1-\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(\frac{\tilde{Q}_1}{Q_2(\alpha_2)}-1\right).$$

Using Eq. (45) (that describes the wealth share after transition for regulated optimists) to substitute for the price decline, we obtain:

$$\tilde{r}_1^f=\rho+g_1-\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right)\frac{1}{\overline{\omega}_1-1}. \tag{A.32}$$

For the benchmark without any prudential policy, following similar steps we obtain:

$$r_1^{f,b}=\rho+g_1-\delta(\eta^*)-\lambda_1^p\frac{1-\alpha}{1-\alpha_2^b}\left(1-\frac{\alpha_2^b\left(1-\beta_2^b\right)}{\alpha(1-\beta)}\right)\frac{1}{\omega_1^b-1}. \tag{A.33}$$

Here, $\alpha_2^b,\beta_2^b,\omega_1^b$ denote the equilibrium variables in the benchmark, which are potentially different than the equilibrium with PMP. In particular, recall from Proposition 5 that $\alpha_2>\alpha_2^b$. Combining this with Eq. (44), we further obtain $\beta_2<\beta_2^b$. PMP decreases the fraction of optimists' wealth held by unregulated optimists, because they react to the policy by increasing their risks more than regulated optimists.

Next note that the wealth shares satisfy the following identity:

$$\begin{aligned}\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right) &= (1-\alpha)\frac{1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}}{1-\alpha_2}\\ &= (1-\alpha)\left(1-\frac{\alpha_2}{1-\alpha_2}\left[\frac{1-\beta_2}{\alpha(1-\beta)}-1\right]\right).\end{aligned} \tag{A.34}$$

Here, the term in the brackets is positive because $\beta_2<\beta$. This identity holds for the pair, (α_2,β_2), as well as the pair, $\left(\alpha_2^b,\beta_2^b\right)$. Combining the identity with the inequalities, $\alpha_2>\alpha_2^b$ and $\beta_2<\beta_2^b$ (which implies $1-\beta_2 > 1-\beta_2^b$), we further obtain:

$$\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2\left(1-\beta_2\right)}{\alpha\left(1-\beta\right)}\right) < \frac{1-\alpha}{1-\alpha_2^b}\left(1-\frac{\alpha_2^b\left(1-\beta_2^b\right)}{\alpha\left(1-\beta\right)}\right).$$

Note also that $\frac{1}{\overline{\omega}_1-1} \leq \frac{1}{\omega_1^b-1}$ since $\omega_1^b \leq \overline{\omega}_1$. Finally, we also have $\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right) < \delta\left(\eta^*\right)$ since $\tilde{Q}_1 < Q^*$. Combining these inequalities with Eqs. ($A.32$) and ($A.33$) proves that $\tilde{r}_1^f > r_1^{f,b}$: that is, PMP sets a higher interest rate than in the benchmark without prudential policies.

Finally, consider how raising the target leverage limit $\tilde{\omega}_1$ affects the interest rate corresponding to PMP. Since raising the leverage limit increases $\tilde{Q}_1$, it also increases the effective depreciation rate, $\delta\left(\frac{\tilde{Q}_1}{Q^*}\eta^*\right)$. Since raising the leverage limit reduces α_2, it also increases β_2 (and reduces $1-\beta_2$). Combining this with the identity in ($A.34$) implies that raising the leverage limit raises the term, $\frac{1-\alpha}{1-\alpha_2}\left(1-\frac{\alpha_2(1-\beta_2)}{\alpha(1-\beta)}\right)$. Combining these observations with ($A.32$) proves that raising the target leverage limit decreases the interest rate, that is: $\frac{\partial \tilde{r}_1^f}{\partial \tilde{\omega}_1} < 0$. In particular, targeting a lower effective leverage limit $\tilde{\omega}_1$ requires setting a higher interest rate, completing the proof. □

A.6. Details of the numerical exercise in Sections 5 and 6

For depreciation, we use the functional form

$$\delta\left(\eta\right) = \underline{\delta} + \left(\overline{\delta}-\underline{\delta}\right)\frac{\left(\eta-\underline{\eta}\right)^{1+1/\varepsilon}}{1+1/\varepsilon} \text{ for } \eta \geq \underline{\eta} \tag{A.35}$$

(and $\delta\left(\eta\right) = \underline{\delta}$ for $\eta < \underline{\eta}$) given some constants $\underline{\delta}, (\overline{\delta}-\underline{\delta}), \underline{\eta}, \varepsilon > 0$. This functional form implies that decreasing factor utilization below the efficient level, η^*, reduces the depreciation rate until $\underline{\eta} < \eta^*$, but it has no effect on depreciation beyond this level. Raising factor utilization above the efficient level increases capital depreciation.

In our numerical examples, we set

$$\underline{\eta} = 0.97,\ \underline{\delta} = 0.04, \overline{\delta} = 0.087, \varepsilon = 20.$$

These choices ensure that the efficient factor utilization and the corresponding depreciation rate are given by [cf. Eq. (16)] with

$$\eta^* = 1 \text{ and } \delta\left(\eta^*\right) = 0.041.$$

In particular, we normalize the efficient factor utilization to one. The choice of $\underline{\eta} = 0.97$ (together with a relatively high elasticity, $\varepsilon = 20$) implies that underutilizing capital by up to 3 percent is not too costly, since it is compensated by a relatively large decline in depreciation. Underutilizing capital beyond this level is much costlier as there is no compensation in terms of reduced depreciation. In our examples, this means that underutilizing capital in the recession is much costlier than underutilizing capital during the boom.

For the discount rate, we set

$$\rho = 0.04.$$

This choice (together with the specification for the depreciation function) ensures that Assumption 2 holds. For the productivity level, we set $A = 1$. This does not play a role as it scales all variables. These choices imply that the efficient asset price level is given by [cf. Eq. (17)]:

$$Q^* = \frac{A\eta^*}{\rho} = 25.$$

For the productivity *growth* rates, we set

$$\begin{aligned} g_3 = g_1 &= 0.1 - (\rho - \delta(\eta^*)) = 0.101 \\ g_2 &= -0.05 - (\rho - \delta(\eta^*)) = -0.049. \end{aligned}$$

These choices satisfy $g_2 < \min(g_1, g_3)$. They also imply that, with no state changes or belief disagreements and if capital were perfectly utilized, then the (risk-adjusted) return to capital would be equal to 10% in the boom and the recovery states and -5% in the recession state [cf. (18)]. In particular, the transition from the boom to the recession represents a 15% shock to asset valuations.

For beliefs, we set

$$\begin{aligned} \lambda_1^o &= 0.09 \text{ and } \lambda_1^p = 0.9 \\ \lambda_2^o &= 4.97 \text{ and } \lambda_2^p = 0.49 \end{aligned}$$

(and $\lambda_3^o = \lambda_3^p = 0$). These beliefs satisfy Assumption 1: compared to pessimists, optimists assign a smaller probability to a transition from boom to recession but a greater probability to a transition from recession to recovery. When combined with the remaining parameters, these values satisfy Assumptions A1-A2, the regularity conditions we need in order to obtain an equilibrium with a positive interest rate in the boom state and a zero interest rate (and suboptimal asset price level) in the recession state.

Recall that we also need Assumption A3 (which is a regularity condition) to ensure that there is a unique equilibrium when optimists' leverage limit binds (cf. Appendix A.3). This condition depends on the equilibrium price function in the recession, $Q_2(\alpha)$. Figure 9 plots the price function corresponding to the parameters described above. We verify that, when combined with the remaining parameters, this function satisfies Assumption A3.